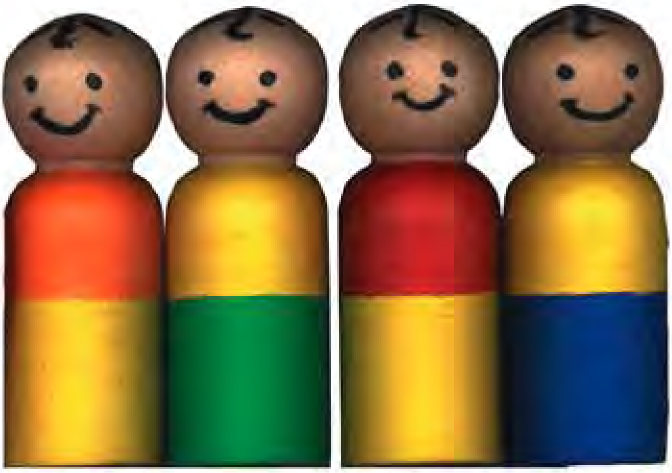

CONSTRUCTION GAMES with KEPLER'S SOLID

Gerhard Kowalewski
David Booth

Parker Courtney

Construction Games with Kepler's Solid

Gerhard Kowalewski

Translation by David Booth
Includes

Rules and Tools

A Translator's Appendix

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Contents

Translator's Introduction

Der Keplersche Körper und andere Bauspiele [1938] appeared in the bibliography of H. S. M. Coxeter's classic treatise *Regular Polytopes* [1947]. Stephen Baer noticed it there, obtained a copy, took an interest in the idea of decomposing polyhedra into building blocks and listed Kowalewski's booklet in the bibliography of the *Zome Primer* [1969]. *Zome Primer*, a manual of new ideas for icosahedral architecture, was an outstanding example of individualistic, do-it-yourself literature.

Eventually the *Zometool* was redesigned, quasicrystals discovered, fullerenes appeared and a tide of articles on icosahedral geometry meant that the pentagonal system in space was no longer an endangered species.

Finally I read a copy of Kowalewski's essay and found that its central interest was *settlement games*, a forgotten precursor of the matching rules that have been widely discussed since the discovery of Penrose tiles. There is more to the icosahedral system than just theorems on non-periodicity and matching rules, interesting as these are. The first mathematical studies of non-periodicity came from an unsuccessful attempt to advance "mechanical mathematics" and were later propelled by the search for a localized, atomistic explanation of quasi-crystals. Kowalewski gives us wider horizons beyond these mechanistic motives, a delightful world of homemade geometrical paperweights, and of Chicky Leberecht crowing his excitement from the rooftop.

I have added an appendix with suggestions, comments and pictures that will save you some time if you keep them in mind while you are reading Kowalewski's essay.

I regrettably neglected to record the names of the Waldorf school students who helped with the translation. I am particularly indebted to them for showing me how to make things more intelligible to young readers.

I have never seen the picture of Shirley Temple playing with the set of MacMahon's blocks that were stolen from Professor Kowalewski. Let me know if you find this picture in some dusty old magazine from a flea market bin.

David Booth
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Austin, Texas

Kowalewski's Foreward

The main object of this booklet is to build a model of Kepler's Solid that is bounded by thirty plane rhombi; it can be found in the polyhedra studies of that great astronomer. It is contructed out of two sets of multi-colored blocks having ten blocks in each set, joined together along identical colors, similar to MacMahon's cubes. This gives a new kind of puzzle that is associated with Kepler's thirty-sided solid; we hope it will attract interest because of its difficulty. A person who would play successfully without knowing the theory would have to be lucky.

There is a connection between Kepler's thirty-sided solid and a construction in six dimensions that is made up of squares and that can be taken as the prototype of the Kepler polyhedron. We are convinced that Kepler would have been filled with enthusiasm to have known that this six-dimensional cube shelters two such Kepler polyhedra within it.

I want to express my debt to my brother, the Königsberg philosopher and an excellent scholar in the theory of color, for his many valuable suggestions. I was able to speak with him during his visit to Dresden in the summer of 1937 about the details concerning the publication of this booklet.

As with the previous booklets of the series that I have established, the aim is to reach a general understanding of the subject.

Dresden, White Stag, Winter 1937/38
Gerhard Kowalewski

Chapter One: Constructions with Multi-Colored Squares and Cubes

A square is divided by its diagonals into four right-angled triangles; let us color them with four different colors (*red*, *yellow*, *green*, and *blue*, for example). These colors are indicated in the figure below by *1*, *2*, *3*, and *4*.

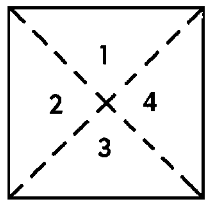


Figure 1

There are six arrangements of colors; distributions of color that are alike except for a mere rotation of the square are not counted as distinct.

Figure 2 shows the six possible arrangements of the colors *1*, *2*, *3*, and *4*.

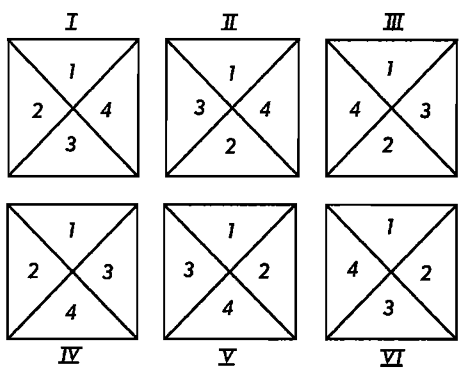


Figure 2

If we had at our disposal a single specimen of each of these six multicolored tiles, then we could take up the following problem.

Build one big square out of four of these multi-colored squares so that the large square has matching colors along the top, bottom, right, and left sides. For example, we might duplicate the edge coloring of Figure 1.

The tiles should be laid so that they obey what we shall call the *domino rule*, that is, squares should only be joined along edges of the same color.

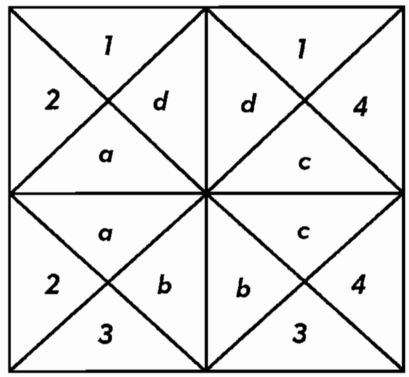


Figure 3

The solution of the problem is easily obtained by looking at Figure 3, where the unknown colors are indicated with the letters *a*, *b*, *c*, and *d*.

It helps here to use the old mathematical trick of naming the unknown. In the lower left-hand tile, *a* appears with the numbers 2 and 3; so it must be different from them. A glance at the upper left tile shows that it must also be different from *1* and 2. So *a* can only be the color *4*. For similar reasons *b* must be the color *1*; *c* must be 2; and *d* can only be 3; so that the whole construction looks like what is shown in Figure 4.

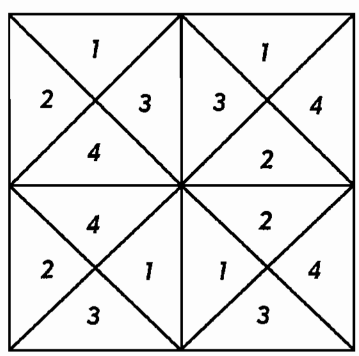


Figure 4

This procedure produces a big square having the same boundary colors as those in Figure 1, using the four tiles displayed in Figure 5. Colors that stand opposite each other in Figure 1 are neighbors in each tile of Figure 5. This observation fully determines the choice of the four tiles.

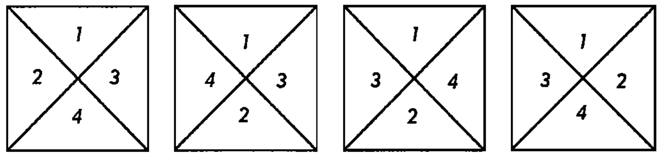


Figure 5

By turning them so that color number *1* is at the top, *3* will have to be either on the right or on the left. Either choice for the placement of 3 yields two choices for the placement of 2 and *4*. With the same four tiles one can obtain a big square, obeying the domino principle, which shows the same edge colors as square VI in Figure 2.

Instead of coloring the quarter squares, one could just as well color along the sides of the squares, framing them in color. The domino principle requires that only edges of the same color may be joined together.

MacMahon's building blocks imitate these problems but in three-dimensional space. In place of squares there are now cubes whose faces are painted in six different colors.

We shall represent these colors by *1*, . . . , 6; as before, we shall regard two arrangements the same if they can be obtained from each other merely by rotating the cube.

Now, how many different color arrangements are there? Suppose that the bottom face, upon which the cube is standing, is colored with the color *1*. One of the colors

2,. . . , 6

must show on top. This gives five different possibilities. These five types depend on whether *1* and 2, *1* and 3, *1* and 4, *1* and 5, or *1* and 6, are opposite each other. The various possibilities are shown on the following page.

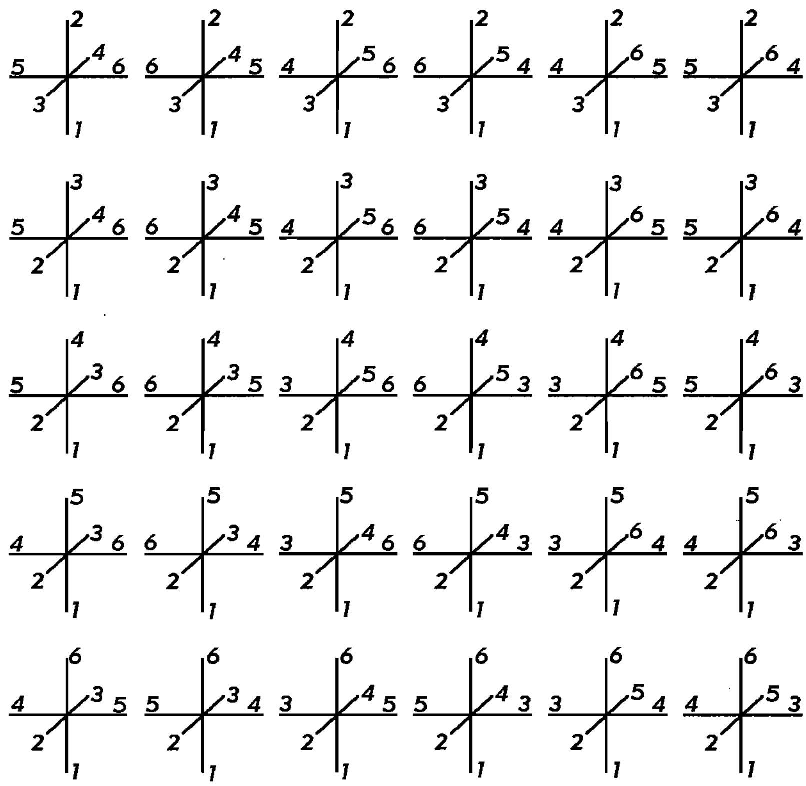


Figure 6

In the type in which *1* and *a* are found on the top and bottom respectively, one can always turn the cube so that the color *b* comes to the front. That leaves three possibilities for the color on the back side. The two remaining colors can be played in two ways (left and right, or right and left). Each of the five types therefore has six multicolored cubes. The total number of MacMahon cubes is 30, while there were only 6 different kinds of bordered squares.

In Figure 6 the 30 MacMahon cubes are shown schematically as crossed axes. Each of the six arms of these crossed axes reaches from the mid-point of the cube to the middle of a cube face and is marked with the color number of the cube face in its direction.

MacMahon's problem is this: Build a big cube using eight of the thirty cubes while adhering to the domino principle. This big cube must show a prescribed color distribution agreeing with a model cube that was previously chosen.

If the pattern cube is, for instance, the first one shown in Figure 6, then the eight cubes must show a pattern that can be exploded as in Figure 7 to make it easier to see.

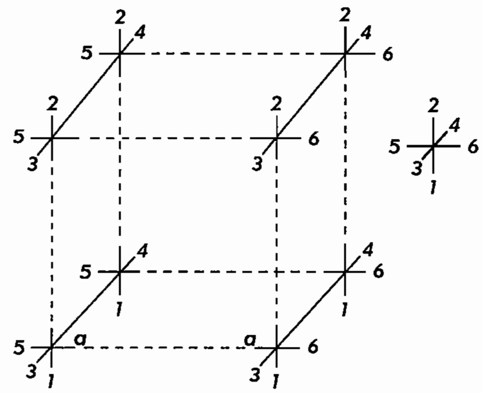


Figure 7

The color a in Figure 7 must be different from *1*, *3*, *5* and also *1*, *3*, *6*. There remain only the possibilities $a = 2$ and $a = 4$. When one of these two possibilities is chosen, the remaining colors are obviously fixed, and one comes to Figure 8 and Figure 9.

To make it easier for you to check your own drawing, we have shown the order in which the absent colors are found, through the use of subscripts. For example, in Figure 8 it is 6_1 that is found first using the fact that this color must be different from *1*, *4*, and 5. Next 4_2 is found using the fact that among *1*, *2*, *3*, *5*, and 6 only *4* is missing, and so on.

MacMahon's problem has two solutions (Figures 8 and 9). By close observation one finds that in both cases the same eight cubes are used.

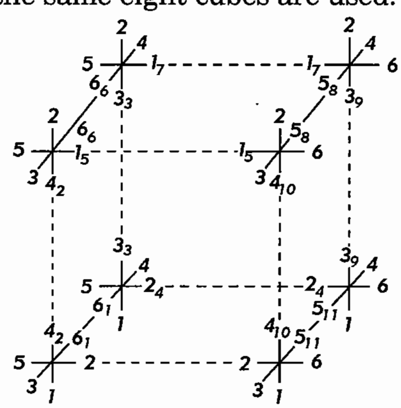


Figure 8

The four cubes on the upper story of one figure build the first floor of the other one.

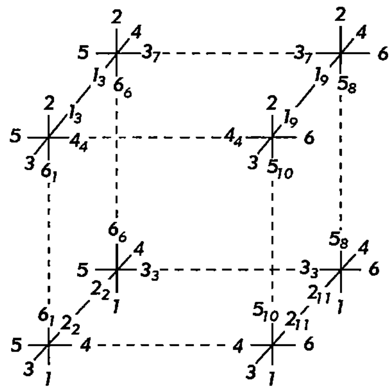


Figure 9

I propose using a form of the crossed axes that is quite handy for building. In contrast to MacMahon's blocks these crossed axes have the advantage that it is easy to control obedience to the domino principle. I glue six cubes of the same size onto the faces colored *1*, . . . , *6* of the MacMahon cube; they are the same size as the cube face to which they have been attached. Figure 10 shows such a cross in which, for the sake of clarity, the front and back arms are left off and only suggested by extended edges.

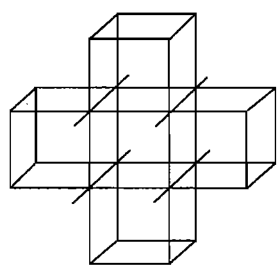


Figure 10

The first specimen of this very decorative toy was lost when I sent it to a toy trade fair. It seems that the thirty multi-colored crosses could not resist the pressure of publicity and entered into the toy trade quite on their own. A picture of Shirley Temple playing with them even appeared in an illustrated magazine.

For further information about MacMahon's blocks one can refer to my book, *Alte und neue mathematische Spiele*, and also to the beautiful monograph of Ferdinand Winter, Dr. of Engineering, on the multicolored cubes, both published by B. G. Teubner. The mathematical, particularly the group theoretic, aspect of MacMahon's problems have been treated by my ingenious student Walter Stams in an illuminating way in one of the latest editions of *Deutschen Mathematik.*

A two-dimensional analog of my crossed axes of glued cubes is shown in Figure 11 on the next page. The central square is mirrored in all four of its sides to produce a cross.

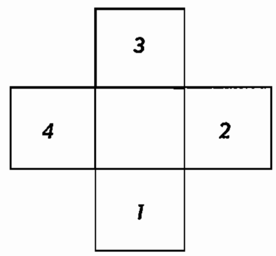


Figure 11

Chapter Two:
Rhombic Skirts for the Platonic Solids

The five Platonic solids – tetrahedron, cube, octahedron, dodecahedron, and icosahedron – were investigated by the ancient Greek mathematicians and constitute the subject of Euclid's *Elements*.

These five solids acquired an unanticipated significance in Kepler's *Mysterium Cosmographicum* (1596). He believed that with their help he could establish a law of planetary distances for the planets known at that time: Saturn, Jupiter, Mars, Earth, Venus, and Mercury. The astronomer associated a sphere with each of these planets; it was centered at the sun and passed through its respective planet. One might wonder whether or not the planets keep the same distance from the sun, an idea that Kepler himself settled by discovering that the planetary paths form an ellipse with the sun at one focus. Kepler found in 1596 that the five Platonic solids could be inserted between the six planetary spheres, so that each such solid has its circumscribed and inscribed planetary sphere. The specific sequence is: Saturn, *cube*, Jupiter, *tetrahedron*, Mars, *dodecahedron*, Earth, *icosahedron*, Venus, *octahedron*, Mercury. If we inscribe a cube into the Saturn sphere, so that its corners are on the Saturn sphere, then the sphere inscribed in that cube (the sphere that touches the cube's faces) is the Jupiter sphere. If you now inscribe a tetrahedron inside the Jupiter sphere, then the sphere that is inscribed into that tetrahedron is the Mars sphere. This construction of Kepler agrees so closely with the facts that we can understand Kepler's enthusiasm and the bombastic style of his publication. Kepler's bold construction was the impulse for the discovery of the small planet Ceres by Piazzi in Palermo in the early nineteenth century. Later when Uranus and Neptune entered into the circle of planets this series attracted less and less interest.

Kepler's planetary construction was very fruitful in stimulating his geometrical research. He did extensive work in the field of polyhedra studies and discovered, for example, the star polyhedra. He had another very good idea too. He created what I call the *rhombic skirts* for the five Platonic solids. Let us fix the idea of this construction using the example of the cube.

From directly above the middle of each of the six faces of the cube one can drop a perpendicular line to the cube edges having some length, say h. Next, consider the cube edge AB. There are two cube faces that meet there. Connect A and B to the mountain peaks E, F situated along the perpendiculars. Doing this creates triangles EAB and FAB. If h is given the appropriate value ($h = a/2$), the triangles will lie in a single plane and will form a rhombus braced by AB along its diagonal. When we construct the rhombus that belongs to each one of the twelve edges of the cube, then we have a rhombic dodecahedron: We shall say that it is obtained by dressing the cube in its *rhombic skirts*.

It is obvious in Figure 12 that $EF = a\sqrt{2}$, $AB = a$, so that the edges of the original cube are the short diagonals of the rhombic dodecahedron. The long diagonals form the edges of an octahedron. The same rhombic dodecahedron serves as the skirts for both a cube and an octahedron.

The rhombic skirts of an icosahedron and those of a dodecahedron are the same things too. These skirts fit together as thirty rhombi making Kepler's celebrated thirty-sided polyhedron; we call it *Kepler's Solid*; it is also known as the *rhombic triacontahedron*.

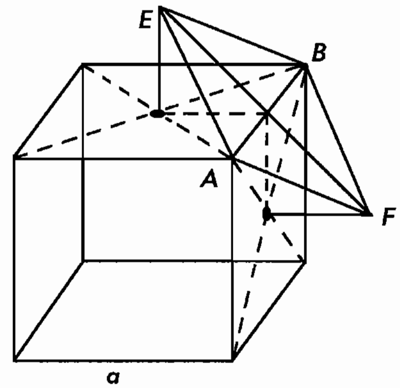


Figure 12

We will see later that the diagonals of the rhombi introduced here are related to each other as a golden section, that is they stand in the ratio

$$\tfrac{1}{2}(\sqrt{5} - 1) \text{ to } 1.$$

This thirty-sided polyhedron was of extraordinary interest to Kepler because two of them can be used to hold the Earth's sphere in his planetary construction. The short diagonals of the thirty rhombi of Kepler's Solid are the edges of a regular dodecahedron, and the longer diagonals are those of an icosahedron. Speculation about the golden section, which Kepler called the *divine section*, exercised a special charm for investigators who were predisposed to mysticism. The ancient Greek mathematician Eudoxus (400 to 350 B.C.) was the first to divide a segment according to the golden section. We make use of this division when, for example, we construct a regular decagon. To find the edge length of an inscribed decagon one needs to divide the radius of a circle in a golden section. The larger piece gives the edge of the decagon. The golden section is that division of a line segment such that the whole piece is related to the larger part as the larger part itself is to the smaller. Taking the whole segment as unity and z as the larger part, the smaller part will be $1 - z$, so the following proportion holds:

$$1 : z = z : 1 - z.$$

From this it follows that $z^2 = 1 - z$ or $z^2 + z = 1$, giving $(z + \tfrac{1}{2})^2 = \tfrac{5}{4}$, that is to say $z = \tfrac{1}{2}(\sqrt{5} - 1)$. This is, therefore, the length of the larger part of a golden section division when the entire segment has a length equal to one. If we develop the irrational number $\tfrac{1}{2}(\sqrt{5} - 1)$ as a continued fraction, we write

$$z^2 + z = 1.$$

$$z=\frac{1}{1+z}=\frac{1}{1+\frac{1}{1+z}}=\frac{1}{1+\frac{1}{1+\frac{1}{1+z}}}=\ldots$$

From this it follows that z, the irrational number of the golden section, can be represented by the infinite continued fraction

$$\frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{1+\ldots}}}}$$

It is the simplest possible continued fraction because it involves only the number 1.

When we draw a circle of radius 1, the continued fraction given above will represent the side of an inscribed, regular decagon. The sequence of convergants

$$\frac{1}{1},\qquad \frac{1}{1+\frac{1}{1}},\qquad \frac{1}{1+\frac{1}{1+\frac{1}{1}}}$$

$$\text{or}\quad 1,\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{5},\ldots$$

shows the following rule, which is based on the general principles of the theory of continued fractions: In each fraction the numerator is the sum of the two preceding numerators and the denominator is the sum of the two preceding denominators. For example, 2 = 1 + 1, 3 = 2 + 1, and 3 = 2 + 1, 5 = 3 + 2, and so on. On the basis of this rule one can easily calculate the sequence of convergants. After 1, 1/2, 3/5, 8/13, ... follow 5/8, 8/13, 13/21,... The number z lies between any two successive terms of this sequence. The fractions of odd index,

$$1,\ \frac{2}{3},\ \frac{5}{8},\ldots,$$

converge from below while those of even index,

$$\frac{1}{2},\ \frac{3}{5},\ \frac{8}{13},\ldots,$$

converge from above. It is a peculiar feature of the sequence

$$1,\frac{1}{2},\ \frac{2}{3},\ \frac{3}{5},\ \frac{5}{8},\ \frac{8}{13}$$

that each denominator appears as the numerator of the next term of the sequence. First you calculate the denominators, q_1, q_2, q_3, ... , by putting $q_1 = 1$, $q_2 = 2$ and then by using the rule $q_n = q_{n-1} + q_{n-2}$. Once that is done the numerators p_1, p_2, p_3, ... of the fractions

$$1,\ \frac{?}{2},\ \frac{?}{3},\ \frac{?}{5},\ \frac{?}{8},\ \frac{?}{13},\ \ldots$$

can immediately be written down using the rule $p_n = q_{n-1}$. This gives the fractions 1, 1/2, 2/3, 3/5, 5/8, 8/13 , ...

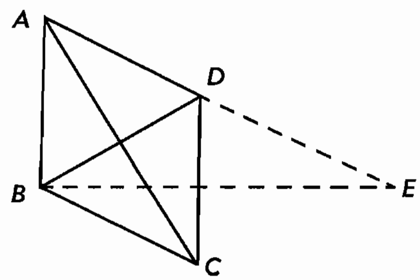


Figure 13

In Figure 13 a Kepler rhombus is shown, that is to say a rhombus whose diagonals *BD* and *AC* stand in a golden section relationship to each other as the side of a regular decagon to the radius of its circumscribed circle. If one were to draw a circle having *AC* as its radius, then the span *BD* would go around the circumference exactly ten times. We will now derive a property of the sharper of the two angles in the Kepler rhombus.

Obviously

$$\tan(\alpha/2) = z$$

where z is the number $\frac{1}{2}(\sqrt{5} - 1)$.

From this follows

$$\tan\alpha = \frac{2\tan(\alpha/2)}{1-\tan^2(\alpha/2)} = \frac{2z}{1-z^2}.$$

Because $z^2 + z = 1$, we also must have that $1 - z^2 = \tau$, thus

$$\tan(\alpha) = 2.$$

To construct a Kepler rhombus having a given side, *AB*, we will erect a line perpendicular to *AB*, say *BE*, having double the length of *AB* (Figure 13). Since $\tan(\alpha)$

= 2, *A* will remain an angle of *ABE* of measure α. Now make *AD* = *AB* and swing two circles about *A* and *B* with a radius *AB*; these will intersect at *A* and also at a new point, *C*. The Kepler rhombus that we are looking for is *ABCD*.

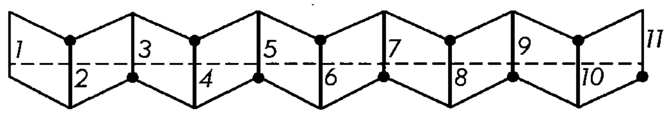


Figure 14

It is vitally important that we have a cardboard model of Kepler's Solid at our disposal. You can easily produce one for yourself. In Figure 14 there are ten articulated Kepler rhombi that can be drawn on cardboard. You must cut out the figures and score the edges that are shown as dark lines in order to fold them more comfortably. It is to be done so that in the finished position the points *1*, *2*, *3*, ..., *10*, *11* form a regular decagon. Then the Kepler rhombi are inserted at the points marked with heavy dots. Finally, we must attach a cap made of five Kepler rhombi to the top and bottom to close the openings. This finishes the construction of Kepler's Solid, the rhombic triacontahedron. You should interrupt your reading at this point in order to finish the model without being hasty in the work.

Let us add a remark concerning the rhombic skirts of the tetrahedron. It can be made plain from Figure 15 that this gives a cube. The rhombi in this case turn out to be squares.

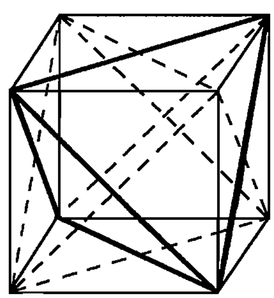


Figure 15

The same cube could equally well be taken as the rhombic skirts of a second tetrahedron shown by dashed lines in Figure 15 and can be said to be the opposite of the one drawn with solid lines.

Coloring Kepler's Solid

We will suppose that you have the cardboard model that we have urged you to construct. If you place it on one of its thirty rhombi, then at the very top there is a rhombus that is horizontal and directly above the base rhombus. If you then turn the model so that the long diagonals of these two rhombi are running from front to back and the shorter ones are running left to right, then one of the thirty rhombi stands in front, one on the very back, another on the extreme right, and one on the extreme left. The planes of these four rhombi, along with those of the top and bottom ones, form a cube with the base, as before. Label these six rhombi all with the number *1*. Next, put an unlabeled face on the bottom, and look for the five other faces that go with it. These are all assigned the label 2. Proceeding in this way until we reach the number 5, all thirty rhombic faces of Kepler's triacontahedron become labeled with numbers. The six rhombi that make up a cube have the same number.

Now you should obtain five colored sheets of paper. Cut six Kepler rhombi that are of the same size as those on the model of the Kepler Solid out of each sheet. Glue a red piece, for example, onto all faces with the label *1*; those numbered *2* get, say, yellow pieces glued onto them; those numbered *3* get green; all with a *4* get blue; finally, make the ones numbered 5 white. In this way the Kepler Solid gets colored with five colors and presents a surprisingly beautiful appearance.

The Kepler Solid has, resulting from its skirt relationship to the dodecahedron and the icosahedron, 20 corners that have three edges and 12 of the five-way corners. Surrounding each of these five-way corners the five colors *1*, ..., 5 are arranged in a specific way. Reading clockwise, beginning with white, we get 12 different arrangements of the other colors and thus of the numbers *1*, ..., *4*. On the model that is here in front of me, I can determine the following 12 different arrangements:

1 2 3 4	*2 3 4 1*	*3 2 1 4*	*4 1 2 3*
1 3 2 4	*2 4 1 3*	*3 1 4 2*	*4 2 3 1*
1 2 4 3	*2 1 3 4*	*3 4 2 1*	*4 3 1 2*

Each of these arises from the numbers *1*, *2*, *3*, and *4* by an even number of transpositions, that is, exchanging the places of some pair of numbers. If one interchanges two of the numbers within the twelve arrangements, *1* and 2 for example, then twelve completely new orderings arise, which (disregarding cyclic changes in the listing) are as follows:

1 4 3 2	*2 3 4 1*	*3 2 1 4*	*4 1 2 3*
1 3 2 4	*2 4 1 3*	*3 1 4 2*	*4 2 3 1*
1 2 4 3	*2 1 3 4*	*3 4 2 1*	*4 3 1 2*

This now exhausts all 24 arrangements of *1*, 2, 3, and *4*. One can conclude from this observation that Kepler's triacontahedron can be colored in two and only two ways with the colors *1*,..., 5 so that the six rhombi belonging to the same cube share the same color. A coloring that can be made to agree with another by a rotation is not taken as different. This fact was observed by my brother in his work on color arrangements in the *Berichten der Wiener Akademie 11*, May 1916.

It is very attractive to see both of the colorings together in front of you. One would have to create two copies of Kepler's Solid. If you are willing to take the trouble, you would acquire an attractive desktop ornament and a valuable model to illustrate group theoretic relationships. You would then want to have a friendly mathematician to teach you some group theory.

Each of the 20 three-way corners of the Kepler triacontahedron is colored with three colors, *a*, *b*, and *c*. There are ten of these triples showing the five colors. Each triple appears twice on opposite three-way corners, but the opposite triples have their colors in different cyclic arrangements as in:

a *b* *c*	*a* *c* *b*

Chapter Three:
The Building Blocks of Kepler's Solid

If you examine the Kepler triacontahedron in five colors, certain building blocks stand out on their own. All you have to do is to look at a corner at which three edges meet to recognize a *stubby block* that is embedded in the Kepler Solid. Three rhombic faces of this block meet at the three-sided corner at their obtuse angles, so that they present the appearance given here.

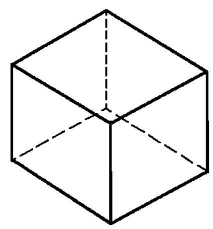


Figure 17

We have indicated the back sides by dashed lines in Figure 17, so that the diagram shows the parallelepiped as it appears. The colored rhombic faces of the parallelepiped that are on the outside surface of the Kepler Solid show three different colors. We will use the same color for opposite rhombi, so that the whole block is colored. It will have three of the five colors, *1*, *2*, ..., *5*, distributed among its rhombic faces so that opposite faces agree. Ten triples can be formed from *1*, *2*, ..., *5* – they are listed below – so there must be ten distinct colored blocks of this kind.

1 2 3	*1 2 4*	*1 2 5*	*1 3 4*	*1 3 5*
1 4 5	*2 3 4*	*2 3 5*	*2 4 5*	*3 4 5*

These ten stubby blocks are, as will be shown, the building blocks for the Kepler Solid.

In addition to these, there exist ten other parallelepiped-shaped building blocks that do not stand out as clearly in the completed model. These have a pair of opposite corners in which the acute angles of the same rhombi meet. We call these building blocks the *steep blocks*. They will be colored in the same way as the stubby blocks are.

We will see that the Kepler Solid can be built up from these 20 colorful pieces, the ten stubby blocks and the ten steep ones, under the rules of the domino principle, so that the outside surface shows the same pattern of colors as the original. It cannot be shown that Kepler himself recognized these facts in detail. We have to assume, however, that he had these 20 building blocks out of which the Kepler Solid can be built. The coloring and the use of the domino principle came later.

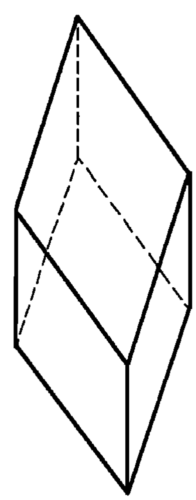


Figure 18

Please interrupt your reading again to make these 20 building blocks out of cardboard with colored paper glued onto them. The construction of the Kepler rhombus has already been explained. The acute angle of such a rhombus fits into a right triangle whose adjacent side is a and whose opposite side is $2a$.

This property is the basis of the construction. If you make this angle the vertex angle of an isosceles triangle, you have half of the Kepler rhombus and you can obtain the fourth corner by swinging arcs at the base vertices of the isosceles triangle whose radius is the length of one of the legs.

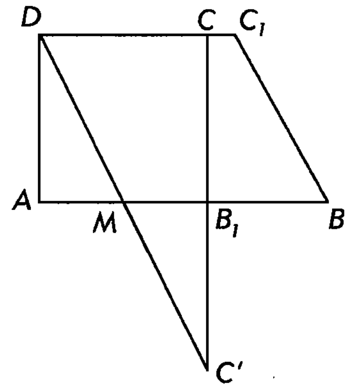


Figure 19

Here is a different way to proceed with the construction. Extend one side of the square $ABCD$, say CB, until its length is doubled (Figure 19). If you draw DC' and make $DC_1 = DM$, then DMB_1C_1 is a Kepler rhombus. It is good to do the construction carefully so that the stubby and steep blocks do not have gaps between them when you use them to build up the Kepler Solid.

If you have never glued cardboard models and do not know how to make a parallelepiped, you can examine Figures 25a and 20b. The first one gives a net for the steep blocks and the second one a net for the stubby blocks. These nets have to be drawn with tabs on cardboard, cut out, scored, and folded along the dark lines. Finally, edges having the same numbers are joined and glued along the tabs.

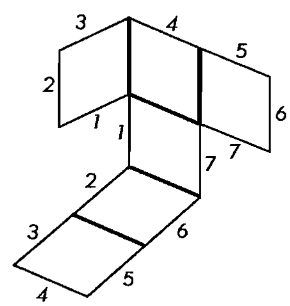


Figure 20a

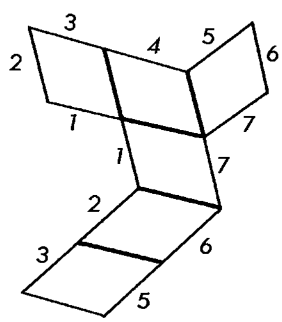


Figure 20b

When you have constructed the ten stubby blocks and the ten steep blocks and have colored every block with its color triple – using red, yellow, green, blue, and white – then you will have a pretty toy in your hands. You can build many interesting structures with it

Keep the domino principle in mind. If you make even more copies of these blocks, then the possibilities will multiply. However interesting the puzzles that arise with these blocks, we do not want to take the space here for them. Besides, it is not at all bad to leave you with some things to work out for yourself.

How to Construct the Five-Way Corners

If you take the colored model of the Kepler Solid in hand and look down at a five-way corner, you see, reading from the right, five colors *1*, *2*, *3*, *4*, *5*. This corner can be built with the help of five steep blocks. If you want to make it comfortably, you need to make a supporting base out of cardboard, which will fit on the corner of Kepler's Solid like a cap.

You also need a cardboard cylinder with both ends open to hold the saucer. A glass can be used instead. The plan for the saucer is shown at the left in Figure 21 and its supporting cylinder on the right. Put the saucer, having the surface *1*, *2*, ..., *5*, on the top opening of the cylinder. Rest five steep blocks having the colors *1*, *2*, ... , *5* on their undersides in the saucer. They will show the very same colors on top, and they must be joined according to the domino principle. You can now test what we say and build everything yourself.

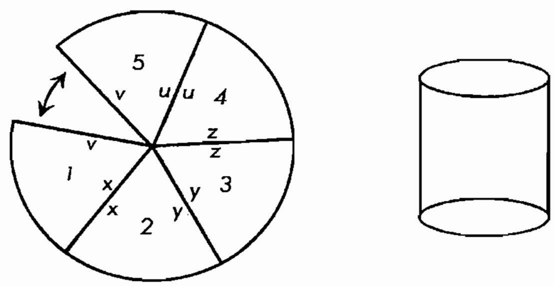


Figure 21

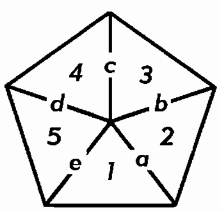


Figure 22

The appearance will vary depending on which colors are used to join the five steep blocks.

In Figure 22 a general scheme is suggested showing what can occur. The block in area *1* of the saucer and the block on area *2* join together along the color *a*, the block on *2* and the one on *3* with the color *b*, etc.

The single blocks carry the color triples listed here.

1 e a   *2 a b*   *3 b c*   *4 c d*   *5 d e*

Thus the permutations of *a*,...,*e* submit to the following conditions:

a	is different from	*1* & *2*
b	is different from	*2* & *3*
c	is different from	*3* & *4*
d	is different from	*4* & *5*
e	is different from	*5* & *1*.

This offers the following possibilities:

a	*b*	*c*	*d*	*e*
3	4	5	1	2
4	5	1	2	3
5	1	2	3	4

The five blocks make a cycle, so you must see the table as wrapping around so that the columns with *a* and *e* at the top are neighbors. You will notice by inspecting the table that each column has a number in common with both of its neighboring columns. These values emerge when we have decided on the ordering of *a*, *b*, *c*, *d*, *e*. Column *a* has a *4* in common with its neighbors *b* and *e*; *b* has a *5* in common with *a* and *c*; *c* shares a *1*; *d* the number *2*; *e* the number *3*. Because of this, we will place the sequence *a*, *b*, *c*, *d*, *e* right after *4*, *5*, *1*, *2*, *3*. The five steep blocks, which we want to use for the production of the observed corner of Kepler's solid, carry the following color triples:

1 3 4   2 *4* 5   3 5 *1*   *4 1* 2   5 2 3.

If you number the five faces of the base *1*, 2, 3, *4*, 5 going counterclockwise, each number makes a triple with the pair of numbers that is opposite it on the pentagonal base.

Let us identify the numbers *1*, 2, 3, *4*, 5 with the colors *red*, *yellow*, *green*, *blue*, *white*. Search through the inventory of blocks to find the ones that have the following color triples. These are the blocks needed to build the corner that we are observing.

red	*green*	*blue*
yellow	*blue*	*white*
green	*white*	*red*
blue	*red*	*yellow*
white	*yellow*	*green*

From this you will be able to build the desired corner easily while resting it on the base. Observed from above it resembles a five-pointed star with the color sequence *red*, *yellow*, *green* , *blue*, *white*, reading counterclockwise.

Around the edges, guided by the domino principle, you can fit in five stubby blocks. The third color will be determined by the opposite arms of the stars that we have made.

For example, between the steep blocks,

red, *green*, *blue*, and *yellow*, *blue*, *white*

lies the stubby block, *green*, *white*, *blue*;

between

yellow, *blue*, *white* and *green*, *white* , *red* is the stubby block *blue, red, white*;

between

green, white, red and *blue, red, yellow*

is *white, yellow, red*;

between

blue, red, yellow and *white, yellow, green*

is *red, green, yellow*;

and between

white, yellow, green and *red, green, blue*

is *yellow, blue, green*.

The colors *red*, *yellow*, *green*, *blue*, and *white* will be cyclically switched at each step. That means each one is replaceable by the one following and the last one by the first one, thus *red* by *yellow*, *yellow* by *green*, *green* by *blue*, *blue* by *white*, *white* by *red*.

Our initial task has been the production of this object made out of the five steep blocks and five stubby blocks. It resembles a cup. Put a rubber band around the whole thing to hold it tight. The rubber band will go across ten vertical edges and the rhombi to which they belong. It takes some skill to put on the rubber band.

Filling the Cup

If you look into the cup, it now offers the possibility of using some of the remaining stubby blocks while respecting the domino rule. You might have already used the *red*, *green*, and *blue* stubby blocks, so that there are only two ways to continue with another stubby block. If you settle on *red*, *yellow*, *blue* no more stubby blocks can be fit in. If you really have done the construction work, you can check this for yourself.

After filling in both stubby blocks – a *red*, *green*, *blue* one and a *red*, *yellow*, *blue* one – we see a hollow in the middle of the cup that invites the insertion of a *red*, *blue*, *white* steep block. On both sides of the inserted block are valleys into which a *yellow*, *green*, *blue* and also a *red*, *yellow*, *green* steep block fit. The next step is the insertion of a stubby block. Indeed, you have a choice between a *red*, *green*, *white* one and a *yellow*, *blue*, *white* one. We will choose the *red*, *green*, *white* one. When this stubby block is added to your construction, you will see clearly a bed for the *green*, *blue*, *white* steep block and, after its insertion, a bed for the last steep block, *red*, *yellow*, *white*. The stubby blocks *yellow*, *blue*, *white*, and *yellow*, *green*, *white* can be accommodated without your having to think about how to do it. This completes the construction of Kepler's Solid.

Six rubber bands are needed to hold the blocks together, because there are six families of parallel edges.

An Exact View of the Construction

If you pay attention to the way that the twenty blocks that make up Kepler's solid come to the outside surface, you can recognize the following in your model: Nine of the ten steep blocks contribute one rhombus to the surface. One steep block lies hidden in the interior of Kepler's solid. Seven of the ten stubby blocks provide three rhombi at the surface. The other three remain hidden inside.

The twenty building blocks of Kepler's Solid have, in total, $20 \times 6 = 120$ rhombi. Thirty of them supply the surface of the solid. The ninety others must, in pairs, make up the interior walls of the construction. Therefore, each interior wall is doubled. There are thus a total of 45 inner walls that come apart onto the 20 pieces that make up the solid body. If you put a window in each rhombus that lies on the outside surface, the nine steep cells have only one window each, and the seven stubby cells have three windows each. Sixteen cells would therefore take in light from the outside. Four cells, one steep and three stubby, would not get any daylight.

The six bands around our construction make a triangle around each three-sided corner of Kepler's Solid and a pentagon around each five-sided corner. The rubber bands cut off triangles and pentagons at the corners. Kepler's Solid has twelve corners with five edges and twenty corners with three edges, so our rubber bands make 12 pentagons and 20 triangles. If we consider projecting the solid out from its center onto a circumscribed sphere, we obtain a partition of the sphere into 12 pentagons and 20, triangles the edges of which are all of equal length. They make great circles that intersect each other at thirty points.

You can easily create this partition on a rubber ball if you draw a great circle and then divide it into ten equal pieces. The ten pieces of this great circle will have equilateral triangles alternating up and down. The new sides of these triangles give us five more great circles. You really need only the triangles that point up. If you extend the left upper sides into the great circles of which they are a part, you will easily obtain a partition of the sphere. It is even easier to circumscribe circles around neighboring upwards and downwards pointing triangles. This way you get ten of the twelve pentagons and without difficulty obtain the parts of the partition that were missing. If you color the six great circles with different colors, a beautiful six-colored model appears on the rubber ball. The following observation is of importance. You can arrange for each band to lie alternately above and below the other at their junctions. You can test for yourself that this rule can be realized without any inconsistency, if you are willing to construct for yourself or to obtain such a colored ball. It hardly needs to be said that these Kepler balls are something new and offer a surprisingly pretty appearance.

Incidental Remarks on the Kepler Ball

If you indicate the six colors that appear on the great circles of Kepler's ball with the numbers *1* through 6 so that each of the 12 pentagons shows five numbers in cyclic order, then the pentagon on the opposite side will have the same numbers taken cyclically but in the reversed order. Each triangle shows a triple taken from *1*,..., 6, while the opposite triangle has the same triad but in the reverse order. Therefore, on the whole ball you can read off six cycles of five numbers and ten cycles of three numbers. On my model I find the following five-fold cycles

(†)
24356, 14536, 14625,
12563, 13246, 12345.

and the three-fold cycles

(∗)
123, 125, 136, 145, 235,
234, 246, 256, 345, 124.

Twenty triples can be created out of six numbers. Ten of these have been selected here. If you create the complement of a triple consisting of the other three numbers, you will find

(∗∗)
456, 346, 245, 236, 235,
156, 135, 134, 126, 124,

thus the missing triples appear.

The ten triples that were selected form an *antipode-free system* [this expression is due to Arnold Kowalewski]. Two triples that together exhaust all six numbers are said to be *antipodal.*

The ten distinguished triples in the list marked with a star above have the following property, which can also be established for (∗∗) as well. We note that each triple $a\ b\ c$ includes the ordered pairs

$$a\,b, \qquad a\,c, \quad \text{and} \quad b\,c.$$

In total there are thirty ordered pairs. These are all of the ordered pairs that can be created out of *1*,..., 6, each written down twice. Because of this property we call (∗∗) a *Steiner triple system of the second order*. A Steiner triple system of the first order is called simply a *Steiner triple system*. The Steiner triple systems will not be used here. Such a Steiner triple system would have to consist of five triples within which all 15 ordered pairs made up of *1* to 6 are contained. If *1*, 2, *3* is one of these triples, there would have to be another that contains the pair *1*, *4*. We can use 5 as a name for the new element that is different from *1*, 2, 3, and *4*. Besides *1*, 2, *3* and *1*, *4*, 5 there cannot be any triple missing. The pair *1*, 6 has to be included, but the third element cannot be among the numbers 2, 3, *4*, 5. We arrive at an impossibility. It is not possible to build a simple Steiner triple system out of *1* to 6, but as the example (∗) shows, there is a Steiner triple system of the second order.

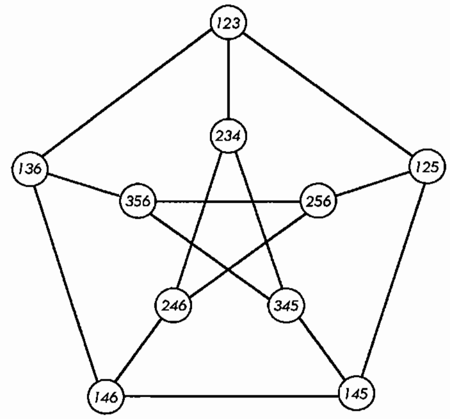


Figure 23

In Figure 23 you can see the triples of the list (∗) distributed into two pentagons whose corners are related using connecting lines. One is an ordinary pentagon, and the other is a pentagonal star. From each triad there are pathways leading to three other triads and indeed just to those triads that share an ordered pair in common.

The system of cycles (†) stands in a simple relation to the triple system (∗). Each round contains five triples that consist of one element and its two neighbors in the cycle. The entire six cycles deliver 30 such triples, and indeed they are the triples (∗) written down three times each.

It is interesting to ask: How many different Kepler balls are there? Switch two elements in the list (∗), 1 and 2, for instance. At the same time switch the two elements that are combined into triples with 1 and 2. In this case it is 4 and 5, so this gives

125, 123, 256, 234, 246,
145, 146, 136, 345, 356.

Therefore, it is the triple system (∗), only in a different order. The transposition that switches two elements a and b is usually expressed by the symbol (ab). If a', b' are two other elements, which in (∗) appear as an ordered pair embedded in some triple, then this system remains invariant under the transposition of elements (ab) and also $(a'b')$. In all there are 15 such permutations, namely

(1 2) (3 5),	(1 3) (2 6),	(1 4) (5 6),	(1 5) (2 4),	(1 6) (3 4),
	(2 3) (1 4),	(2 4) (3 6),	(2 5) (4 6),	(2 6) (4 5),
		(3 4) (2 5),	(3 5) (4 6),	(3 6) (1 5),
			(4 5) (1 3),	(4 6) (1 2),
				(5 6) (2 3).

With the help of the permutations in the first row you can bring any of the elements 2, 3, 4, 5, 6 into the first position in place of 1, without altering the system (∗). When 1 is held fixed, this brings to our attention the following transpositions:

$$(2\ 4)\ (3\ 6), \quad (2\ 6)\ (4\ 5), \quad (3\ 4)\ (2\ 5), \quad (3\ 5)\ (4\ 6), \quad (5\ 6)\ (2\ 3).$$

With the help of these permutations you can steer any of the elements 3, *4*, 5, *6* into the position of 2. Once *1* and *2* are fixed, there remains only the permutation (3 5)(*4 6*); that is, you can exchange 3 and 5 in the triples they form with *1* and 2, and also exchange *4* and *6* without changing the system (∗).

Are there other permutations of the numbers 3, *4*, 5, *6* that leave the system (∗) unchanged? Since *3* and 5 and also *4* and 6 have to be permuted with each other, it only remains to check whether (3 5) and (*4 6*) separately will leave the system unchanged. Neither does leave the system unchanged. For example, (3 5) and (*4 6*) act on the triple *1 3 6* to give *1 5 6* or *1 3 4* respectively: These are triples from the system (∗∗). Apart from the identity, which leaves everything in place, there is only one permutation of 3, *4*, 5, *6* that leaves the system (∗) unchanged, namely (3 5)(*4 6*). Altogether there are 1·2·3·4 = 4! = 12 different permutations that can be carried out on the numbers *3*, *4*, *5*, *6*, so there are 12 systems that arise from permutations of its elements. We have made use of a theorem of group theory that has been known for a long time.

You can check what has been done by thinking as follows: Along with the ordered pair *1*, *2* the additional pair *3*, *4* or 3, *5* or 3, *6* or *4*, *5* or *5*, *6* will appear within a triple. That makes six possibilities. Each time there are two different systems that are transformed into each other when the elements that are in triads with *1* and 2 are switched with each other. For example the system (∗) changes into the following new system when *3* and 5 are interchanged:

$$\begin{array}{lllll} 1\,2\,5, & 1\,2\,3, & \underline{1\,5\,6}, & \underline{1\,3\,4}, & 2\,4\,6, \\ \underline{2\,4\,5}, & 1\,4\,6, & \underline{2\,3\,6}, & 3\,4\,5, & 3\,5\,6. \end{array}$$

The new triples are underlined.

The answer to our question, therefore, is that 12 different Kepler balls can be created with six colors.

Chapter Four: Thirty Little Men and Kepler's Solid

Upon reflection a lot can be said about the Kepler Solid that we made out of twenty blocks and six colors and then bandaged together in order to hold the building blocks in place. We already recognized that each band could be arranged in a handy way with crossings going alternately over and under (on each of the rhombic faces – you will need a model in hand – two bands cross, one on top and one beneath). Let us chose the colors

black, red, yellow, green, blue, white

for these bands. On each rhombus, one color is on top and one on the bottom. The top color is the color of the band that crosses above and the bottom color is the color of the band that falls beneath. If you name the colors *1, 2, 3, 4, 5,* and *6*, each rhombus has a top number *a* and a bottom number *b*. This gives an ordered pair *a b*. Thirty ordered pairs can be formed from the numbers *1* to *6*. Any of the six numbers can occupy the first position; after this is fixed, any of the remaining five can occupy the second position. These thirty ordered pairs of six objects are distributed on the thirty rhombic faces of Kepler's Solid. On neighboring faces, that is, those sharing a common edge, there are pairs of the form *a b* and *c a*, in which *a*, *b*, and *c* represent three distinct terms of the sequence *1, 2, 3, 4, 5, 6*. The band that passes over the common edge of these two rhombi is *a*. It is on the top at one face and on the bottom at the other. The two ordered pairs have, as we see, an element in common but this element takes the first position in one ordered pair and the second position in the other; this weakens the domino character of the neighborhood. You could call *ab*, *ac* or *ba*, *ca* a *strong domino junction* and *ab*, *ca* a *weak domino junction*. The thirty ordered pairs made out of sequence *1, 2, 3, 4, 5, 6* are distributed over the rhombi of Kepler's Solid so that neighboring pairs stand at a weak domino junction. This distribution will be given more attention. My brother, the founder of *systematic color theory*, an extraordinary deep theory with rich connections to practical questions, calls this distribution a *settlement*.

You can pose a *settlement problem* whose solution is found in the reasoning given above: The thirty ordered pairs formed from sequence *1, 2, 3, 4, 5, 6* have to be distributed on the faces of Kepler's Solid so that neighboring pairs are connected according to the weak domino principle.

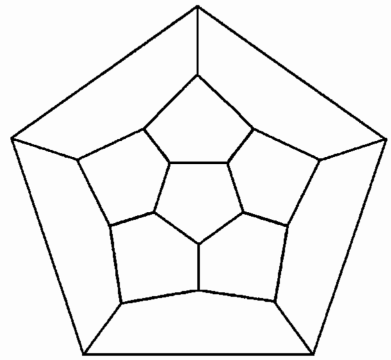


Figure 24

This settlement can be represented in a plane diagram because the resulting distortion does not change the nature of anything. Figure 24 shows a regular dodecahedron that has been pressed flat. The small pentagons lie over the big one like a sheet, and the large pentagon is merged with the edges of the entire figure. If you were to blow this up as if it were an airtight bag, it would pop out as a figure in space resembling a regular dodecahedron. We know that Kepler's Solid consists of the rhombic skirts that are shared alike by the regular dodecahedron and icosahedron. We can do this with a distorted dodecahedron just as well as with a regular dodecahedron, but we have to be satisfied with representing the rhombi as quadrilaterals. Choose a point on each face of the distorted dodecahedron and connect it with the corners. When you take away the edges of the dodecahedron, there will be a distorted Kepler's Solid.

This construction is carried out in Figure 25. The dodecahedron edges are still visible as dotted lines. The point within the large pentagon that makes a basis for the figure has been thrown to infinity. We have to think of straight lines going to infinity from the corners of the large pentagon. It is as though we were representing a sphere by a plane of infinite radius. The point at infinity is the point that is opposite the center point of the diagram.

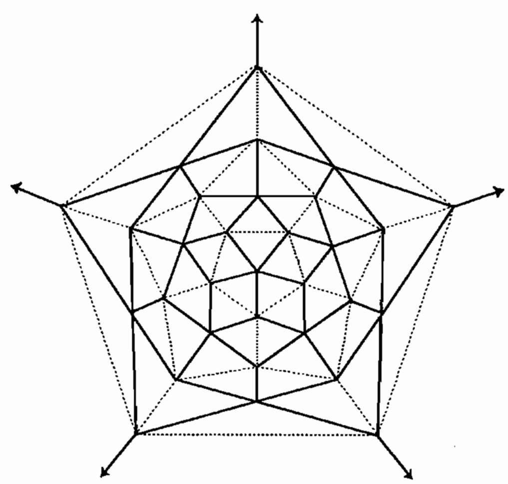


Figure 25

If you pay attention only to the dark lines and think away the dashed lines, you can see the plane divided into thirty quadrilateral areas, from which five lines go out to infinity. These thirty quadrilaterals will suffice for the settlement problem instead of the rhombi of Kepler's Solid. In Figure 26 you can see the ordered pairs taken from sequence *1*, *2*, *3*, *4*, *5*, *6*.

*	*1 2,*	*1 3,*	*1 4,*	*1 5,*	*1 6,*
2 1,	*	*2 3,*	*2 4,*	*2 5,*	*2 6,*
3 1,	*3 2,*	*	*3 4,*	*3 5,*	*3 6,*
4 1,	*4 2,*	*4 3,*	*	*4 5,*	*4 6,*
5 1,	*5 2,*	*5 3,*	*5 4,*	*	*5 6,*
6 1,	*6 2,*	*6 3,*	*6 4,*	*6 5,*	*

They are placed into the thirty regions of the diagram according to the weak domino principle.

If you want to make a game out of this settlement problem, replace the ordered pairs with little men: In the pair *a b*, "*a*" represents the color of his pants and "*b*" the color of his jacket. The thirty little men are to be distributed one on each region of the game board so that no man ever has the same color jacket or the same color pants as his neighbor. There is a common color that makes a kind of neighborhood connection between them but it comes out in different ways. The pants color of one is the jacket color of the other.

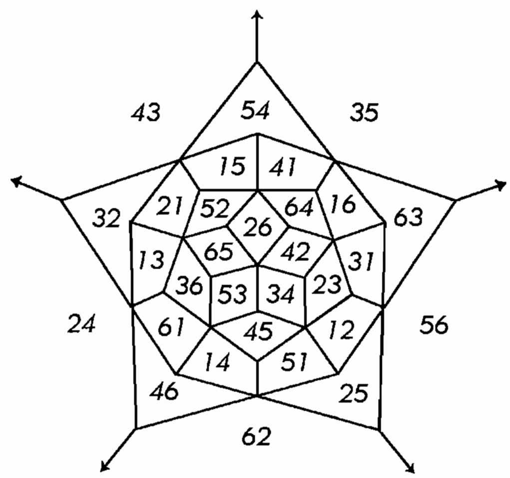


Figure 26

Each little man, looking across his four boundaries, can say with satisfaction that no neighbor is exactly like himself. If he has, for example, red pants and a blue jacket, then two neighbors have red jackets but not red pants and the other two have blue pants but not blue jackets. Every little man is special, different from his neighbors. In the settlement game you can put, for example, two men that do not share a common color down onto regions that meet at a corner. Then, using the principle of weak domino junction, go on with the settlement. In Figure 27 you can see one of these beginnings. Into the region marked with a star, that bounds the regions that have already been used, comes an ordered pair *a b* which is related to both *1 2* and also *3 4* by a weak domino junction. To connect itself in the right way to *1 2*, it has to be either a *1 a* or *2 a*. The pair *a 1* connected in the previously described way to *3 4* when *a* = *4*, and to *2 a* is when *a* = *3*. Therefore, the region ✱ must be filled either with *4 1* or *2 3*. If you choose *4 1*, then the pair *2 4* definitely goes into the region ✱✱. If you choose *2 3*, you will have to fill in ✱✱ with a *3 1*. The same certainty will arise for filling in the region ***. We shall leave it to you to finish the game board settlement. That is the best way to get involved with the idea.

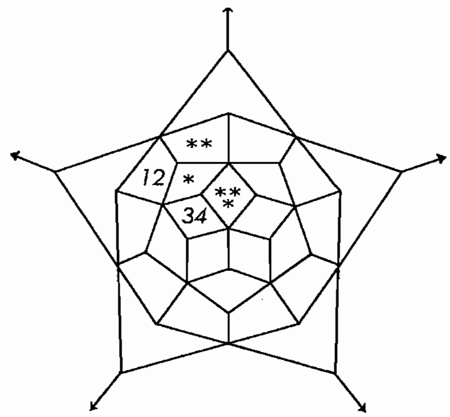


Figure 27

If you are a bold player, you can choose more than a two-fold beginning originally and then see whether you can still play all the little men. Even if you have not used all of them, you still might be pleased enough to show it to someone. The number played can be considered as your score for the game. Two players can take turns trying their luck. Add up the scores to compare the totals at the end.

Anyone who knows the theory can solve the settlement problem if nothing is showing in the beginning. You can solve it easily by picking out a ring of ten regions corresponding to one of the six bands around Kepler's Solid. Each region of one of these rings meets the ring of its neighbors at two opposite edges. In Figure 28 a ring is shown using darkened lines. You can begin to fill the regions of this ring with the pairs 1 2, 1 3, 1 4, 1 5, 1 6. On the opposite sides put 2 1, 3 1, 4 1, 5 1, 6 1. The only thing that can go into the region that bounds 1 2 and 5 1, for example, is the pair 2 5; only 3 5 can touch 5 1 and 1 3, etc. The regions bordering the ring can be settled in this way. Ten of the five-way corners will be missing. An ordered pair is then forced by the demands of the weak domino principle with respect to the pairs that have already been used. You can check this by continuing the columns that have been started in Figure 28.

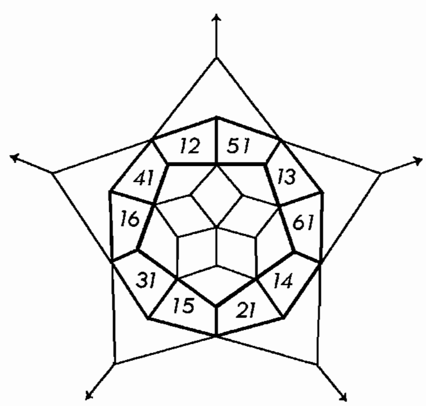


Figure 28

If you consider ordered pairs to be taken from the numbers $0, 1, 2, 3, 4, 5$ instead of $1, 2, 3, 4, 5, 6$; and if you make the number $6A + B$ correspond to the ordered pair $A\ B$, then the numbers that are obtained are all different. It would follow from

$$6A + B = 6A' + B'$$

that

$$6(A - A') = B' - B$$

$B' - B$, a difference of two numbers taken from the sequence $0, 1, 2, 3, 4, 5$, is a multiple of six. That is possible only if $B' - B = 0$. It would follow immediately from this that $A - A' = 0$. As long as the ordered pairs $A\ B$ and $A'\ B'$ are distinct, it cannot be that the equations $A = A'$ and $B = B'$ hold simultaneously. When the pairs $A\ B$ are substituted into the expression $6A + B$, thirty different numbers are obtained. The smallest is $6 \times 0 + 1 = 1$. The largest is $6 \times 5 + 4 = 34$. The numbers of the form $6x + x$, that is 7, 14, 21, 28, the multiples of 7, are not included. The following 30 numbers remain:

1	2	3	4	5	6
8	9	10	11	12	13
15	16	17	18	19	20
22	23	24	25	26	27
29	30	31	32	33	34

Suppose you have a Kepler's Solid that is not colored, and ordered pairs made up of numbers $0, 1, 2, 3, 4, 5$ are spread out over the thirty rhombi according to the weak domino principle. By substituting the number $6A + B$ for the ordered pair $A\ B$, the thirty numbers in the above table will be marked onto the rhombi. Kepler Solids of this kind can be used as dice (to actually use them as dice, they would have to be made out of wood or porcelain). This solid is much closer to a sphere than a cube is. It rolls easily across a table. However it stops, there is a number showing on the top face, which can be read as the outcome of the throw. You could have twelve such dice all numbered differently corresponding to the twelve different Kepler Solids. They originate from the sequence given in the preceding table, and eventually they will coincide. There are many possibilities here for creating games, but we do not want to go into that now. The situation also offers interesting opportunities for making problems in probability.

Chapter Five:
The Rhombic Dodecahedron

The rhombic dodecahedron, as we have learned, can be obtained by putting rhombic skirts on the cube or the octahedron. We will now turn to a construction in which we can see the rhombic dodecahedron as an exception to solid bodies in general.

To get better involved in the situation, we want to treat a simpler construction in a similar way. Consider a parallelepiped, or as we prefer to say a *block*, lying in space. Moving out of one corner of this block are three line segments AB, AC, and AD. If we put them together in pairs – like forces in the construction of the parallelogram of forces – each line segment is brought to the ends of the others by a parallel translation, giving the six line segments BC_1, BD_1, CD_1, CB_1, DB_1, and DC_1. We slide the original line segments to the new endpoints as well and obtain in this way the three edges B_1A_1, C_1A_1, D_1A_1. All twelve edges of the block are derived from the three basic line segments AB, AC, AD. If you project the block onto a plane by a parallel projection (in Figure 29 it is the plane ABD) there arises an image of the block in the plane.

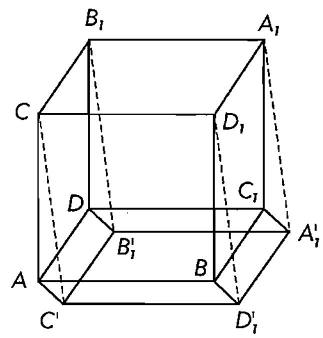


Figure 29

In this plane image everything is derived from the three basic line segments AB, AC, AD in exactly the same way as in the spatial construction. If you leave out the line segments going from A and A_1', there remains the hexagon $BC_1DB_1'CD_1'B$, the projection of the hexagon $BC_1DB_1CD_1B$ consisting of those six edges of the block that do not touch either of the two opposite corners A and A_1.

Using this construction, we can design, in our ordinary space, the image of a four-dimensional block in which we lay down four basic line segments, AB, AC, AD, AE, and take them together in pairs, triples, and then four at a time as shown in Figure 30.

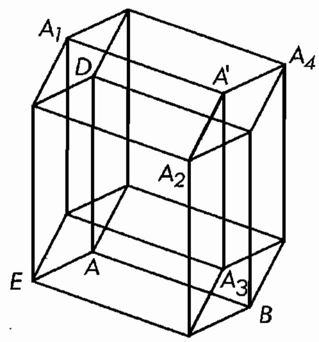


Figure 30

You can carry out the construction so that you get a block out of each triple:

✱	*AC*	*AD*	*AE*
AB	✱	*AD*	*AE*
AB	*AC*	✱	*AE*
AB	*AC*	*AD*	✱

You will obtain four blocks, each having points that are opposite to *A*. Call them A_1, A_2, A_3, A_4 respectively. These points can be translated to give the missing edges, which will all meet at the point *A′*.

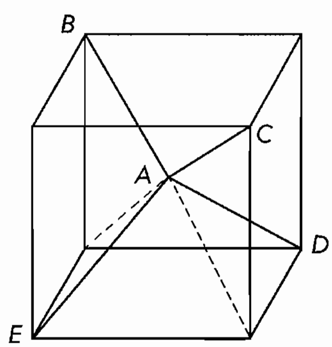


Figure 31

In Figure 31 we see a cube whose center *A* is connected by segments *AB*, *AC*, and *AD* to four of the cube corners. These corners are so chosen that no two have an edge in common. These four named segments are obviously of the same length and between any two of them they form the angle 2α such that $\tan \alpha = \sqrt{2}$. This reminds us of the ancient symbol of the *irminsul*, a very important regular figure of four legs joined together at their ends.

[The irminsul was a wooden object that played a central role in the ancient religion of the Saxons. It was probably destroyed in 772 by the Franks, who defeated the Saxons and forced their conversion in a bloody struggle. Several puzzles surround this object. Its location and religious significance, for example, are matters of conjecture. One mysterious aspect of the matter is that this cross-like tree, and associated representations carved in stone, suggest Christian symbolism as if the religion of the Saxons was somehow parallel to Christianity, even though the Saxons remained beyond the pale of the Roman Empire.

Kowalewski suggests that the arms of the irminsul, like *ABCDE* in Figure 31, did not lie in a plane. – Ed.]

If you use these four legs for the construction of the four-dimensional block, then the points *A* and *A'* will coincide (as in Figure 32). It is a special case of Figure 30.

[The points *ABCDE* in Figure 30 are placed in a general, arbitrary position whereas the corresponding points in Figure 31 are located a specific positions in a cube. – Ed.]

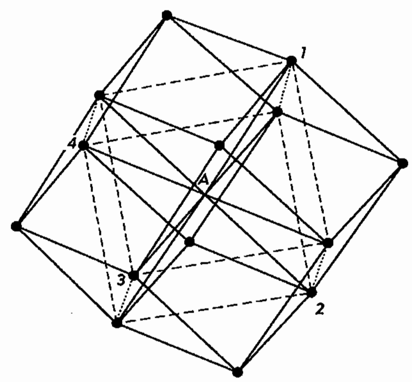


Figure 32

We know the rhombic dodecahedron as the rhombic skirts of the cube. If you leave out the lines that come from the points *A* and *A'*, you have a rhombic dodecahedron in front of you. You may do the same thing in Figure 30, that is, leave out the edges that belong to *A* and *A'*. This would give you a dodecahedron with parallelogram faces. The rhombic dodecahedron we create by putting skirts on a cube is a special case of the parallelogram dodecahedron. These parallelogram dodecahedrons originate in a four-dimensional block by discarding two opposite corners along with their edges; we must project the figure on one of the infinitely many three-dimensional spaces that play the same roll in four dimensions as planes do in three dimensions.

A cube in three dimensions projects onto a hexagon in a plane when you project perpendicularly in a direction through two opposite corners of the cube. Therefore the parallelogram dodecahedron is the spatial analog of the planar hexagon.

You see the parallelogram dodecahedron in the right light when you think of it in connection with a four-dimensional block. We can see it as having its origin in four-dimensional space, from a four-dimensional block whose opposite corners have been removed with their attached edges. The four-dimensional block has sixteen corners. Four edges go out from each corner, uniting in pairs and making six parallelograms. Altogether their are 16×6 parallelograms. Each of them will be counted four times, however, because they have four corners. The total number of these parallelograms in a four-dimensional block is $\frac{16 \times 6}{4}$ or 24. We have left out two opposite corners with their attached edges, so the two times six parallelograms that belong to them will be removed too. The twelve parallelograms that remain appear as the parallelogram dodecahedron.

The Archetype of the Rhombic Dodecahedron in Four-Dimensional Space

The four-dimensional source of the rhombic dodecahedron can be seen by imagining a cube in four-dimensional space and then removing two opposite corners along with their attached edges. Only twelve of the 24 squares of this cube will remain. They present us with the four-dimensional origin of the rhombic dodecahedron. The rhombi were originally congruent squares. If an inhabitant of four-dimensional space looked at this structure from a distance, along the line that connected the two opposite corners that were removed, he would see the rhombic dodecahedron as a three-dimensional figure, thus in an entirely different way from the way that we would see it. This viewer would experience, so to speak, something similar to what we experience when we look at a cube from a distant point along the line connecting two of its opposite corners, with the edges attached to those corners made invisible. The picture that this offers to us is that of a regular hexagon. It is the projection of the remaining six cube edges onto a plane perpendicular to our line of sight.

Since we cannot look into a four-dimensional space as we do our usual space, we need to depend on the analytical methods of Descartes for the treatment of geometrical questions in four dimensions. It is like Braille, which offers a substitute for vision. In ordinary space a point is indicated by three Cartesian coordinates x, y, z in connection with three perpendicular axes. If you want to fix a specific cube, you have only to give the coordinate triples of its eight corners. If, for example, the coordinate triples

0, 0, 0	1, 0, 0	0, 1, 0	0, 0, 1
0, 1, 1	1, 0, 1	1, 1, 0	1, 1, 1

are written down, then you have a cube that is supported by the three positive axes and has edges of length 1.

The corresponding body in four-dimensional space, where there are four rather than three coordinates, has the corners

0, 0, 0, 0

1, 0, 0, 0   0, 1, 0, 0   0, 0, 1, 0   0, 0, 0, 1

1, 1, 0, 0   1, 0, 1, 0   1, 0, 0, 1

0, 0, 1, 1   0, 1, 0, 1   0, 1, 1, 0

0, 1, 1, 1   1, 0, 1, 1   1, 1, 0, 1   1, 1, 1, 0

1, 1, 1, 1.

If you now remove the opposite corners 0, 0, 0, 0 and 1, 1, 1, 1, there remain the corners of the 12 squares that build up the structure that we call the four-dimensional archetype of the rhombic dodecahedron. The endpoints of an edge always have coordinate quadruples that differ in only one position, the other terms agreeing. The corners of a rhombic dodecahedron in our space can be given the 14 labels

	1, 0, 0, 0	0, 1, 0, 0	0, 0, 1, 0	0, 0, 0, 1
(✱)		1, 1, 0, 0	1, 0, 1, 0	1, 0, 0, 1
		0, 0, 1, 1	0, 1, 0, 1	0, 1, 1, 0
	0, 1, 1, 1	1, 0, 1, 1	1, 1, 0, 1	1, 1, 1, 0

Here the endpoints of each edge carry coordinate labels that differ only in one term, thus as little as possible, or as we like to say, they stand in a *strong domino junction.*

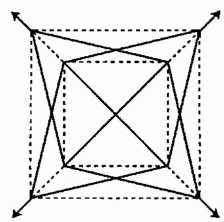


Figure 33

Figure 33 shows a plane representation of a rhombic dodecahedron that comes from the cube shown in dashes. The midpoint of the big square is thrown to infinity, understood as the point towards which all four of the rays in the figure are heading. The 14 corners here can be given the labels from the table (✱) above. The strong domino principle must be in control along each edge.

Whoever knows the origin of this net in four-dimensional space will be able to solve the problem without difficulty. The edges of the original object, which are all edges of a four-dimensional cube, fall into four classes, each made up of six segments parallel to one of the four coordinate axes. The same grouping applies to the lines of the net of Figure 34.

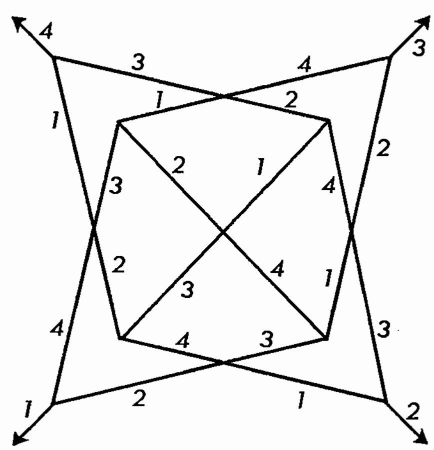


Figure 34

You may realize that in the archetypal, four-dimensional object, edges that go out from one corner belong to different classes and the edges at the opposite point to the same classes. This makes it easy to write down numbers *1*, *2*, *3*, *4* representing these edges on the diagram of Figure 33.

You start by numbering the lines going out from the center of the picture with the numbers *1*, *2*, *3*, *4*. As you proceed with the numbering go along with the rule that opposite edges of quadrilateral regions get numbers of the same class. Figure 34 shows a completed numbering of this kind.

Now you have to realize that running along an edge of class a, that is at the transition from one end to another, only the a^{th} term of the corner label will change, while the other terms remain the same. If you picture this to yourself, you can say that the labels *1, 0, 0, 0* and *0, 1, 1, 1* cannot possibly stand on corners linked by a line of class *1*. The other end of such a line would have to be either *0, 0, 0, 0* or *1, 1, 1, 1*, but these labels are not available. Likewise, *0, 1, 0, 0* and *1, 0, 1, 1* cannot go on corners that are joined by a line number *2*. The corresponding fact holds for *0, 0, 1, 0* and *1, 1, 0, 1*, as well as *0, 0, 0, 1* and *1, 1, 1, 0*. These eight labels correspond to the eight corners at which three edges end.

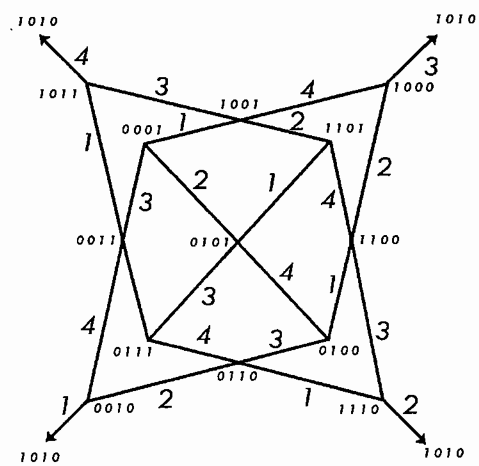


Figure 35

We may partition the labels without any difficulty, as you can test for yourself (Figure 35). If you fix *1, 0, 0, 0* and *0, 1, 1, 1*, as can be done in two different ways, it puts a further constraint on the labels because the numbers of the edges always specify which term must be changed along that line.

When the numbers of the lines are not given, it is not so easy to write down the labels, especially for someone who doesn't know the secret of these net diagrams. To make a game out of it, replace the four termed sequences with little, white towers that have four bands of colored either black or red going around them.

For example, instead of the label *1, 0, 1, 0*, there is a tower having the colors red, black, red, black going in order from the top to the bottom (Figure 36). Start filling the thirteen circles in Figure 37 according to the strong domino principle so that the towers that are connected by lines differ in color on only one band. When thirteen positions are occupied, there will be one tower left over.

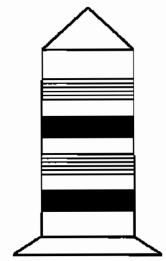


Figure 36

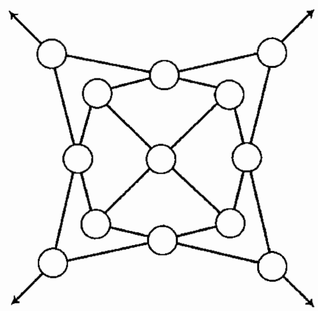


Figure 37

You can set up the game so that this tower, a king without a country, is taken out of the game right in the beginning. If you choose the wrong one, you will not be able to place the towers properly. The tower that is excluded must have two red bands and two black bands. That is the big secret. The game loses its appeal when people know that. The purest pleasure can only be obtained by a childish soul who is unburdened by the theory. When the towers cannot all be placed, you can still give points for correctly played towers.

Four Bands around the Rhombic Dodecahedron

The rhombic dodecahedron is, as we have often pointed out, a parallel projection of a four-dimensional original into a three-dimensional space. Parallel lines remain parallel in the projection. From this it follows that the rhombic dodecahedron in our space has four kinds of edges, each of which consists of six parallel segments in the original. You can verify it with one glance at a rhombic dodecahedron; it shows in Figure 32 too.

Now we want to put two altitudes through the middle point of each rhombus perpendicular to the sides (Figure 38) and give these altitudes the same class number as the sides to which they belong. Six of these altitudes carry the number *1*; six carry the number 2, and so on. The altitudes with the number *a* stuck on them make a zone around the rhombic dodecahedron; an entire class of edges, the class *a*, crosses the zone.

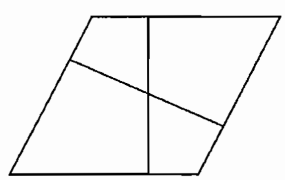


Figure 38

You can substitute four colors, *red*, *yellow*, *green*, and *blue*, for instance, and give the zones a suitable width. If you color the whole rhombic dodecahedron black, it produces a model that makes a strong aesthetic impression. Just as with the zones that surround Kepler's Solid, you can arrange the zones so that they pass alternately over and under each other at their six junctions. On each of the twelve rhombi we have a top zone *a* and a bottom zone *b*. Let us return to using the numbers *1*, *2*, *3*, and *4* and write the number of the top zone in the first position and the number of the bottom zone in the second position, thus making ordered pairs out of the numbers *1*, *2*, *3*, *4*. These ordered pairs

*	*12*	*13*	*14*
21	*	*23*	*24*
31	*32*	*	*34*
41	*42*	*43*	*

are spread over the twelve faces of the rhombic dodecahedron in such a way that the pairs that stand on neighboring faces always have an element in common; the element is in the first position of one pair and in the second position of the other. We call this relation the *weak domino principle*. If you want to carry out a settlement on the faces of the rhombic dodecahedron using ordered pairs with the elements *1*, *2*, *3*, *4* according to the weak domino principle, you can start by putting the pair *1 2* on any of the faces. The adjoining pairs *2 3*, *2 4*, *3 1*, *4 1* will be distributed among the four neighboring faces. These four faces can be seen to have equal status by rotating and reflecting on the rhombic dodecahedron. So the pair *2 3* could go on any of the four faces. Once that position has been assigned, the positions of the remaining three are uniquely determined. This shows how the course of further settlement is forced. The result can be seen in Figure 39. Allowing for rotations, the whole thing has two settlements of the desired kind: Figure 39 and its mirror image. As with Kepler's Solid, you can make dice by replacing the ordered pairs on the faces of the rhombic dodecahedron with numbers.

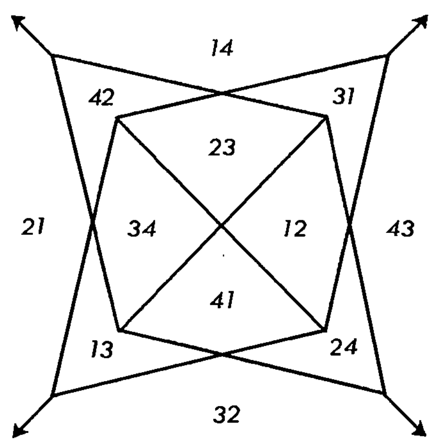


Figure 39

For this purpose it is advisable to use the numbers *0*, *1*, *2*, *3* instead of *1*, *2*, *3*, *4* and to change the pair *a b* into the number $4a + b$. In place of the pairs

*	*01*	*02*	*03*
10	*	*12*	*13*
20	*21*	*	*23*
30	*31*	*32*	*

there are the numbers

*	*1*	*2*	*3*
4	*	*6*	*7*
8	*9*	*	*11*
12	*13*	*14*	*.

You will obtain two dice this way because there are two settlements, not counting rotations. Because there are twice as many faces as there are on the dice of the usual kind, games played with rhombic dodecahedral dice offer more excitement.

If you project the four zones that we put around the rhombic dodecahedron out from the center of the solid onto a circumscribed sphere, four band-like great circles arise. They divide the sphere into six quadrilaterals and eight triangles. You may paint these bands on the sphere in such a way that each of them goes alternately over and under at the six junctions. The actual construction, on rubber balls for instance, will not present you with difficulties. The two balls of four colors that are obtained will give all the more satisfaction because they cannot be bought anywhere.

The fourteen regions of such a sphere partitioned into six quadrilaterals and eight triangles correspond to the corners of a rhombic dodecahedron. The previously considered settlement of the four-termed labels in the corners of the solid with four-termed labels consisting of zeroes and ones allows us to see the partitioning of the sphere into triangles and quadrilaterals as a settlement of faces as well.

Construction of the Rhombic Dodecahedron with Four Blocks

If you think back on the construction of the rhombic dodecahedron from regular four-legged pieces and glance again at Figure 32, you will see four blocks out of which the solid rhombic dodecahedron can be made. Any three of the four legs make a solid corner into which one of the four blocks can fit.

You may build these four blocks out of cardboard. They are bordered by three rhombi whose diagonals are in the ratio $1:\sqrt{2}$. The blocks that are involved are clearly the stubby ones.

If you number the legs of one of these four-legged pieces with *1*, *2*, *3*, *4*, you can number the four blocks with the triples

2 3 4   1 3 4   1 2 4   1 2 3.

The block *2 3 4* fits into the solid corner with the legs *2 3 4* so that three of its edges lie along the just-mentioned legs. If you color the legs of the four-legged piece, using for example, *red*, *yellow*, *green*, and *blue*, you can use red to color the edges of the block that are parallel to a red leg; likewise, the edges that are parallel to a yellow edge can be colored yellow, and so on. Carry out this coloring of edges, making sure that the colored strip is large enough that the colors show on both of the adjoining faces. Three edge colors are used on each block. On a block with *red*, *yellow*, and *green* edges, for example, you will paint the edge in stripes that go out from an obtuse corner where three obtuse angles of the rhombi meet. It does not matter in which order you paint the colors, because when you color according to the rule, the colors are cyclically reversed on the opposite corner. The blocks look beautiful when you use black as the background color or glue on black matte paper.

Joining the blocks together to make a rhombic dodecahedron is easy because you know, for example, that the *red, green, yellow* block must rest itself on the *red, yellow, green* leg of the four-legged piece, which, by the way, is not really needed for the assembly. The *red, yellow, blue* block rests on the *red, yellow, blue* leg of the four-legged piece and must fit in such a way that the two yellow and two red edges coincide. All four blocks come together at the origin of the four-legged pieces with obtuse angles. You will find it easy to convince yourself how easy it is to join the four blocks together. To hold the construction together, wrap a rubber band around each set of six parallel edges; the previously mentioned division of the surface into eight triangles and six quadrilaterals stands out clearly. Try to use rubber bands that are colored like the edges that they cross.

Chapter Six: Kepler's Solid Once Again

The strong link between the rhombic dodecahedron and the four-dimensional cube gave us the link between the domino settlement of the rhombic dodecahedron's corners and the four-termed *0-1* labels. This suggests that we reconsider Kepler's Solid, the rhombic triacontahedron, once again in search of similar properties for it.

The structure of the rhombic dodecahedron arises from four systems, each consisting of six mutually parallel edges. If you take the model of Kepler's solid in hand, you will notice that its 60 edges fall into six systems of ten parallel edges. It therefore has some connection with the cube in six-dimensional space. Part of this six-dimensional cube, which we cannot imagine in its true form, will be the archetype of Kepler's solid. It is a difficult matter to decide what part of it needs to be removed. Our experience with the rhombic dodecahedron gives us clues that help us find our way through the darkness that comes from our lack of visual imagination in six dimensions.

We start by providing numbers for the various classes of edges of the Kepler Solid. All parallel edges have the same class number. One such numbering of the edges can be seen in the distorted plane projection shown in Figure 40.

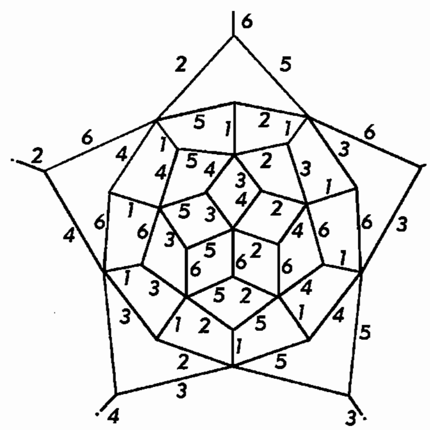


Figure 40

Choose any corner that has five edges and assign these edges the sequence of numbers *1*, *2*, *3*, *4*, *5*. Follow the rule that states that the opposite sides of a quadrilateral, the parallel sides of a rhombus on Kepler's Solid, must be given the same class number. There will be ten sides of quadrilaterals remaining, and they can be assigned the number *6*.

In Figure 41a the corners of Figure 40 are enlarged into little circles. The only exception is the corner that lies at infinity. It would have to be drawn so large that the entire figure would be included within it. The class numbers for the edges of the network are copied from Figure 40. Even though the scale of Figure 41a is enlarged for clarity, some of the class numbers on the short edges at the center of the diagram have been omitted.

Figure 41b shows the interior region of Figure 41a, magnified so that the class numbers of the edges in the center are clearly visible.

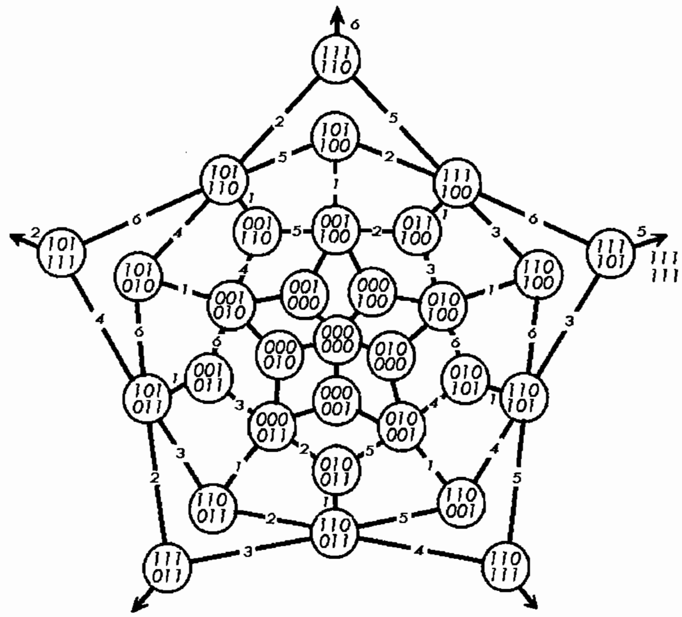


Figure 41a

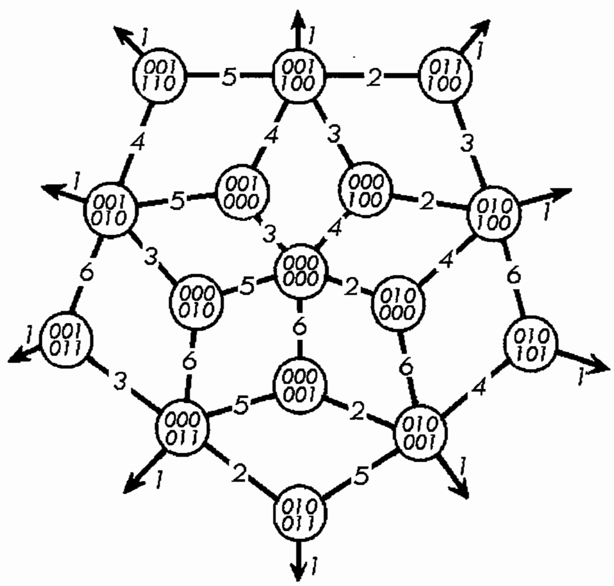


Figure 41a

In the little circles there are 32 different six-fold *0-1* labels written in accordance with the strong domino principle. That is, when an edge carries the number a, the labels differ from each other in exactly one position, the a^{th} term. This shows that such a settlement really exists – it can be carried out consistently. We begin by entering the label *0 0 0 0 0 0* into the center circle from which the strands *2, 3, 4, 5, 6* emerge. To make it more clear, we prefer to use a two rowed symbol

$$\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix}$$

and we shall hold to this for the other labels as well. In traversing a strand with the number a, only the a^{th} term of the six-fold label will be changed. This determines the labels that go into the five neighboring circles. For example, the strand marked *2* changes the second term and leads

$$\text{from } \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} \text{ to } \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{matrix}.$$

In Figure 41a and 41b the settlement was continued in the necessary way until it was completely finished. The infinitely distant point must be assigned the label

$$\begin{matrix} 1 & 1 & 1 \\ 1 & 1 & 1. \end{matrix}$$

We can verify that in the finished settlement a strand of type a actually does have two six-fold labels, differing only in the a^{th} place. The easiest way to check it is to look at the twelve circles having five strands, including the infinitely distant point, to see whether the labels on their neighboring circles are related properly. This will take care of all 60 strands.

The settlement has only 32 labels; that is, only half of the possible ones are used. We provide here a table of the labels that were used in Figures 46a and 46b, ordered according to a principle that is easily seen.

		000000 111111		
010000 101111	001000 110111	000100 111011	000010 111101	000001 111110
010100 101011	010001 101110	001100 110011	001010 110101	000011 111100
011100 100011	010101 101010	010011 101100	001110 110001	001011 110100

This establishes the connection between Kepler's triacontahedron and the six-dimensional cube. The 64 six-fold *0-1* labels are nothing more than the Cartesian coordinates of the corners of such a cube, exactly as the two-fold *0-1* labels

00 10 01 11

represent the four corners of a square, and the three-fold labels

000	*100*	*010*	*001*
011	*101*	*110*	*111*

represent the eight corners of a cube lying in three-dimensional space.

Concerning these 32 labels the following can be said: Five of the six corners that are neighbors of *000000* are chosen; here they are

010000 *001000* *000100* *000010* *000001*.

In the table these entries form the upper part of the second line. Take them in a cyclic sequence.

010000	*000100*	*001000*
|		|
000001		*000010*

Any two of these will form a square with *000000* at one corner. the fourth corner of such a square can be obtained by adding the labels. This gives five more corners that have the following marks:

010100 *001100* *001010* *000011* *010001*.

They stand in the top line of the third row of the table having 32 entries. The operation that we have just carried out is indicated schematically in Figure 42.

In six-dimensional space a "square" is really square, rather than rhombic in shape as shown here. The coordinate labels for the next corners beyond those shown are easy to calculate if you know that edges will have the same length and direction as they did before.

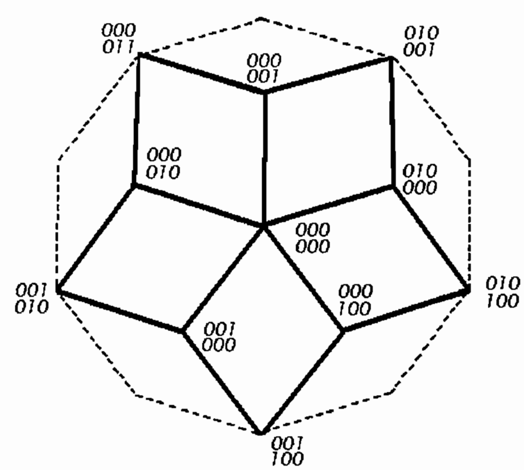


Figure 42

Thus, if

$x_1, \ldots, x_6$ $y_1, \ldots, y_6$ $z_1, \ldots, z_6$ $u_1, \ldots, u_6$

are corners of a parallelogram, then as you walk around the edges – see Figure 43 – you see that the step $x \to y$ and $u \to z$ produce the same coordinate changes.

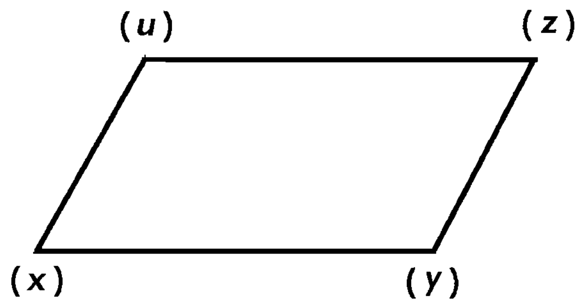


Figure 43

This means that

$$y_i - x_i = z_i - u_i \quad (i = 1, ..., 6)$$

or

$$x_i + z_i = y_i + u_i.$$

If you add the coordinate labels from two opposite corners, you always get the same result. In this way we obtain the five new coordinate labels that are on the boarders of Figure 44 but had not yet appeared in Figure 42. In the table of 32 labels they lie in the upper part of the last row.

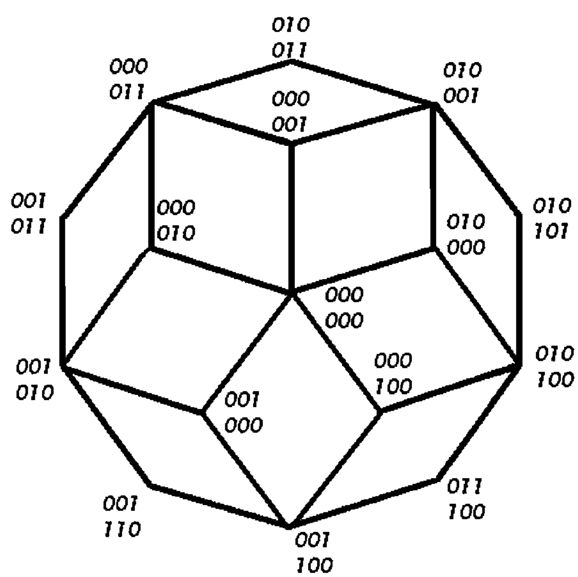


Figure 44

This explains the meaning of the 16 points that stand in the upper half of our double-rowed table. The bottom half shows the opposite corners. In the six-dimensional cube, just as in the bottom half, opposite corners have complementary coordinates. Complementarity reveals itself in the replacement of *0* marks with *1*s and vice versa; you can arrive at a complementary label by switching *0* and *1*.

If you choose 16 corners of the six-dimensional cube, as in Figure 44, and use the operation of complementarity to find the ones underneath, you get all 32 cube corners in the table. When you remove the bottom layer of corners with their attached edges, you have a view of the six-dimensional cube that is the original form of Kepler's Solid.

To better explain Figure 44 in words, we will take a corner, D, which with A and two of A's neighbors, B and C, make a square. Call the resulting quadrilateral $DBAC$. A is framed by its neighboring corners B and C. The rule of the construction runs as follows: We choose five of the six neighbors of the cube corner A_1 and name them A_2, A_3, A_4, A_5, A_6 in any order. In six-dimensional space the neighbors of A_1 are completely independent. In the schematic figure we arrange the subscripts 2, ..., 6 in a counterclockwise cycle just to give a better view of the whole process. Now we build the point sequences $A_6A_1A_2$, $A_2A_1A_3$, $A_3A_1A_4$, $A_4A_1A_5$, $A_5A_1A_6$, into quadrilaterals and name these cube corners A_7, A_8, A_9, A_{10}, and A_{11} (Figure 45).

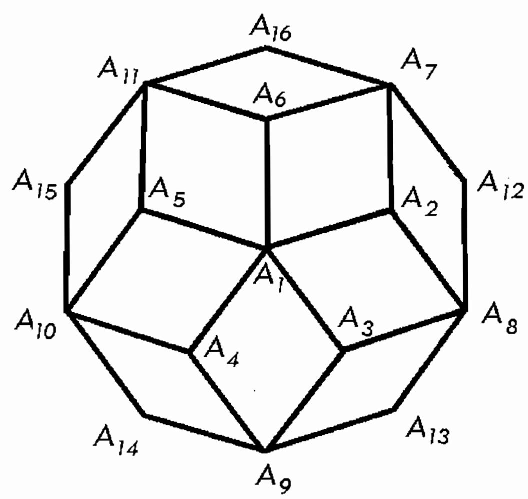


Figure 45

Continuing, we make the quadrilaterals $A_7A_2A_8$, $A_8A_3A_9$, $A_9A_4A_{10}$, $A_{10}A_5A_{11}$, and $A_{11}A_6A_7$. Therefore, we obtain the cube corners A_{12}, A_{13}, A_{14}, A_{15}, and A_{16}. Each of these cube corners has its opposite corner. These opposite corners can be indicated by A'_1 ,..., A'_{16}. The other 32 cube corners together with their attached edges will be pulled out of the cube. The cube will be regarded here as a system of 64 points with certain connecting edges. After picking out 32 corners, together with their edges, there remains an object that has a cap like Figure 50, as well as another cap that is opposite it. There are ten easily calculated edges that join the two caps. If you look and the end points of the edges in Figure 45 and compare them with those in Figure 44, you obtain the labels of the connecting corners on the front cap.

	010100	*011100*	*001100*	*001110*	*001010*
(✱)	*001011*	*000011*	*010011*	*010001*	*010101*

The complementary labels give the corners that are on the opposite cap

	101011	*100011*	*110011*	*110001*	*110101*
(✱✱)	*110100*	*111100*	*101100*	*101110*	*101010*

Each label in (✱) has one and only one neighboring label in (✱✱); indeed the labels in the first row of (✱) are neighbors to those in the second row of (✱✱) and vice versa. If you were to magically change the two complementary labels and hold the new figure above the old one so that complementary marks were arranged perpendicularly over one another, you would need to turn the figure around 180 degrees to get two neighboring corners lying in the same direction from the center.

There can be no doubt that there are more neighbor relationships along the lines built up in Figure 41a of 32 cube corners, counting the distant point. You can dive down into the diagram with two typical labels in order to test the matter. The label *001100* has five neighbors in the figure. The list of thirty-two is only missing the one neighbor *001101*. We see only three neighbors for the label *010101*. The other three – *000100, 011101, 010100* – cannot be found in the list of thirty-two. We have provided a table that gives the missing neighbors for the whole sequence of points. First, there are the twelve labels that have only one missing neighbor.

[The twelve labels appearing in Figure 41 that have one missing neighbor lie in the top row of the table that follows. The missing neighbor of each one appears beneath it. Together they form the first block of two rows in the table. – Ed.]

Beneath twelve labels in the top part of the table stand, in a similar way, the twenty labels that have three missing neighbors. Again the label is on top with its missing neighbors beneath it. The facts can be read off with the help of the strand numbers. The table is on the following page.

000 000	010 100	010 001	001 100	001 010	000 011	111 111	101 011	101 110	110 011	110 101	111 100
100 000	010 110	011 001	001 101	011 010	000 111	011 111	101 001	100 110	110 010	100 101	111 000

010 000	001 000	000 100	000 010	000 001	001 011	001 110	010 011	010 101	011 100
110 000	101 000	100 100	100 010	100 001	011 011	011 100	011 011	000 101	011 000
011 000	011 000	000 110	010 010	001 001	001 111	000 110	010 111	011 101	011 110
010 010	001 001	000 101	000 110	000 101	001 001	001 111	010 010	010 111	011 101

101 111	110 111	111 011	111 101	111 110	110 110	110 001	101 100	101 010	100 011
001 111	010 111	011 011	011 101	011 110	100 100	100 001	100 100	111 010	100 111
100 111	100 111	111 001	101 101	110 110	110 000	111 001	101 000	100 010	100 001
101 101	110 110	111 010	111 001	111 010	110 110	110 000	101 101	101 000	100 010

In addition, we will survey this inventory of neighboring labels to display those that are missing from the list of thirty-two. Labels having two or four number *1*s appear three times; the others each occur once.

LIST OF MISSING NEIGHBORS

		100000 011111		
110000 001111	101000 010111	100100 011011	100010 011101	100001 011110
011000 100111	010010 101101	001001 110110	100010 011101	100001 011110
011010 100101	011001 100110	010110 101001	001101 110010	000111 111000

This is also a list of the thirty-two cube corners and attached edges that were eliminated when we took Kepler's triacontahedron from its archetype, the cube in six-dimensional space. We may think of this six-dimensional cube as being especially perfect because the thirty faces, which are rhombi in three dimensions, are squares there. We have something to say about the neighbor relationship holding among the 32 corners of this cube.

Twelve corners, those that have an odd number of *1*s, have one neighboring corner in the old list, therefore five in the new list. The remaining ones are connected to three in the old list and three in the new. The new list has neighbor relationships like the old one. This leads us to the conjecture that a second rhombic triacontahedron can be created from the new thirty-two points along with their connecting edges. Figure 46 shows the confirmation of this conjecture. We call the second Kepler Solid the *complement* of the first.

From the table showing the neighbor relationship holding between the thirty-two old and the thirty-two new points, it can be seen that there are 72 cube edges passing between the old 32 and the new 32. They fall into six classes containing twelve parallel edges each. These edges radiate from two opposite five-fold corners of Kepler's triacontahedron and from the ten neighbors that have parallel connections with the complementary Kepler Solid.

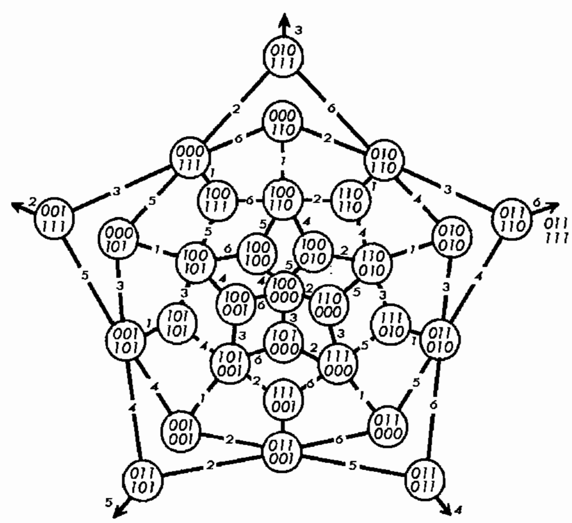


Figure 46

Thirty-Two Corners, Old and New

If you specify the places in a six-fold *0-1* label in which the *1*s can be found, the label is completely determined. Run through the thirty-two old labels and write the number groups on round paper or wooden disks that will serve later as game markers. We associate the disk with nothing written on it with the sequence *000000*. The other number groups that we use are listed on the following page.

2	3	4	5	6
24	26	34	35	56
234	246	256	345	356
156	135	134	126	124
1356	1345	1256	1246	1234
13456	12456	12356	12346	12345
		123456		

If a *0* is changed to a *1* in some *0-1* label, then the corresponding number groups are expanded by one term. On the other hand, changing a *1* to a *0* causes a term in the number group to fall away. We will designate two such number groups as neighbors if a term has been dropped from the longer one. Two neighboring groups consist of an r-termed number sequence and an $(r + 1)$-termed one in which the shorter one forms a constituent part of the longer one. Here are the number groups that correspond to the new thirty-two corners.

		1		
12	13	14	15	16
23	25	36	45	46
235	236	245	346	456
146	145	136	125	123
1456	1346	1245	1236	1235
3456	2456	2356	2346	2345
		23456		

To compare these number groups with those of the old thirty-two, we openly display the previously discussed complementarity. We see that both systems supplement each other in a complete catalog of the number groups that can be built from *1*, *2*, *3*, *4*, *5*, *6*.

1 2 3 4 5 6

12 13 14 15 16 23 24 25 26

34 35 36 45 46 56

123 124 125 126 134 135 136 145 146 156

234 235 236 245 246 256 345 346 356 456

1234 1235 1236 1245 1246 1256 1345 1346 1356

1456 2345 2346 2356 2456 3456

23456 13456 12456 12356 12346 12345

123456

Settlement Games on the Kepler Net

The Kepler net is made of corners and edges of Kepler's Solid in planar representation as seen in Figure 41a, where the distant point is associated with one of the corners. The corners usually show as nodes and the edges as strands of the net.

Take out the 32 game markers on which the number groups of the thirty-two old corners are written, including the blank marker. To play, you put away one of the markers – it corresponds to the infinitely distant point – and then you try to settle the rest of the markers, step by step, on the net. Neighboring nodes must be covered with markers whose number groups are related by crossing out one of the numbers in the longer of the two groups. We know that such a settlement can be done consistently, but someone who does not know the theory would have to be lucky to do it. Interesting games should not be merely matters of chance.

The infinitely distant point in the Kepler net has five neighbors. We are in a position to identify the number groups of this type from our previous investigations. Using Figure 46a, we can list them; in addition to the blank number group they include the following:

24 26 34 35 56
1356 1345 1256 1246 1234
123456.

These are just the number groups that have an even number of terms (0, 2, 4, or 6). The player who does not know this and puts some other marker away to represent the infinitely distant node will end up in a disaster no matter how well he plays otherwise. If we use the thirty-two game markers that belong to the number groups of the thirty-two new corners, then the nodes that have five strands will be associated with the number groups that have an odd number of terms. We saw this in Figure 51. If we reserve a marker with an even number of terms for the infinitely distant point, the settlement can never be completed.

On a game board with either the thirty-two old markers or the thirty-two new ones, two players can take turns playing markers after one is put aside to represent the infinitely distant node. The one who settles the most nodes would win.

A Very Difficult Settlement Problem

We can produce sixty-four game markers that show all the possible number groups made out of the numbers *1*, *2*, *3*, *4*, *5*, *6*, including the blank marker. These markers can completely settle the two forms of the Kepler net according to the neighbor principle. This simultaneous settlement can be seen by translating the labels in Figures 46a and 51 into number groups. Anyone who does not know our theory would have to be very lucky in order to complete this difficult settlement problem. Perhaps the extraordinary difficulty of this problem will be especially fascinating for people who have a feeling for these things. It does not need to be stressed that this game is entirely new.

If we think about the course of this investigation, it would not be immodest to say that our new game came out of six-dimensional space. The mathematician can produce rare things that are seldom seen. He can import games from the sixth dimension. My heartfelt enthusiasm for this fantastic idea reminds me of the fairy tale in which Chicky Leberecht was drinking tea in his rooftop coop while romanticizing about the caravans that brought this fine tea all the way from China.

We should not let the topic pass without saying that instead of using number groups written on game markers, we could produce white, six-story towers with red and black bands around them. Towers standing on neighboring nodes must differ in color on only one band; the other bands must agree in color.

Finally, we could employ markers that do not show the *0-1* labels but, letting x_1, x_2, x_3, x_4, x_5, x_6 represent *0*s and *1*s, show the numbers $32x_1 + 16x_2 + 8x_3 + 4x_4 + 2x_5 + x_6$ instead. This correspondence assigns numbers between 0 and 63 to the *0-1* sequences. Restricting ourselves to the old set of thirty-two cube corners the assignment gives us the sequence

0, 1, 2, 3, 4, 8, 10, 11, 12, 14, 16, 17, 19, 20, 21, 28,
35, 42, 43, 44, 46, 47, 49, 51, 52, 53, 55, 59, 60, 61, 62, 63.

The neighbor principle requires that connected nodes be occupied by numbers that differ by a power of two.

If we want to use only the thirty-two labels from the sequence 0 to 63 without having gaps, we must first pull out six terms from the middle having the numbers 29, 30, 31, 32, 33, 34. The numbers 28 and 35 now stand on the ends of this gap. Left of 28 is a six-fold piece 22, 23, 24, 25, 26, 27. On the right of 35 is the six-fold piece 36, 37, 38, 39, 40, 41. Farther to the right and left we pull out pieces to create four gaps of one element each. Blocks of three terms, then two terms, then one, and then three terms are left standing. Going to the left, 21, 20, and 19 are left standing; farther yet, 17 and 16; then 14 is left; finally 12, 11, 10 remain. The terms 18, 15, 13, and 9 are scratched out. To the right there remain 42, 43, 44; going farther, 46, 47, and then 49; finally 51, 52, 53 stand, while 45, 48, 50, and 54 fall out. At the ends we produce two three-fold gaps going left from 8 and right from 55. Figure 52 shows the gaps in 0, ..., 63 in schematic form.

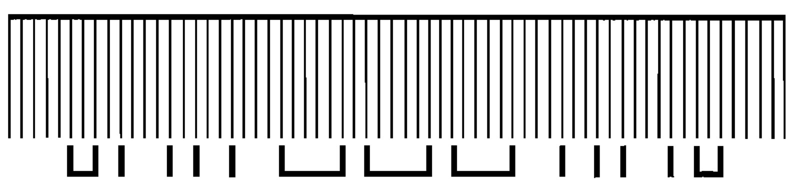


Figure 47

The Building Blocks of Kepler's Solid in Six-Dimensional Space

The twenty building blocks that we used to make Kepler's triacontahedron are congruent three-dimensional cubes in six-dimensional space. It is not difficult to follow the construction using coordinates in six dimensions. We will do this very briefly. The first step is to choose a corner to be *000000* and pick out its five neighboring corners

010000 001000 000100
000001 000010.

It forms a cyclic sequence the way it is written here. If we denote the corners by

B C D
F E,

and if we give the names *A* and A_1 to *000000* and *100000* respectively, we have built a three-dimensional cube out of *AB*, *AC*, AA_1. Likewise there are cubes made from *AC*, *AD*, AA_1, out of *AE*, *AF*, AA_1, and out of *AF*, *AB*, AA_1. Let the sequence *G*, *H*, *I*, *K*, *L* be the fourth points of *ABC*, *ACD*, *ADE*, *AEF*, and *AFB* as given schematically in Figure 48.

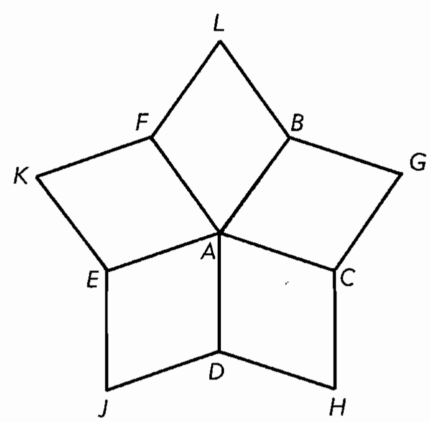


Figure 48

In six-dimensional space parallel edges radiate out from *B*, *C*, *D*, *E*, and *F*. We call them AA_1, BB_1, CC_1, DD_1, EE_1, FF_1. The process continues until we make a complete three-dimensional cube from these edge triples.

BL	*BG*	BB_1
CG	*CH*	CC_1
DH	*DI*	DD_1
EI	*EK*	EE_1
FK	*FL*	FF_1

This gives us twenty blocks already. By using our former construction in three-dimensional space, we can find the twenty blocks that are missing.

It would be advantageous to move out from the corner A_1 that is surrounded by eight blocks. We will indicate vectors here by the use of bold, lower case letters.

Out of *A* go six edges of the six-dimensional cube, which we call $\mathbf{e}_1, \ldots, \mathbf{e}_6$. Going out of A_1 we construct the three dimensional cubes

$$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \quad \mathbf{e}_1\mathbf{e}_3\mathbf{e}_4 \quad \mathbf{e}_1\mathbf{e}_4\mathbf{e}_5 \quad \mathbf{e}_1\mathbf{e}_5\mathbf{e}_6 \quad \mathbf{e}_1\mathbf{e}_6\mathbf{e}_2$$
$$\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4 \quad \mathbf{e}_2\mathbf{e}_5\mathbf{e}_6 \quad \mathbf{e}_2\mathbf{e}_5\mathbf{e}_4$$

Then, going from $A_1 + \mathbf{e}_2$, $A_1 + \mathbf{e}_3$, $A_1 + e_4$, $A_1 + e_5$, $A_1 + e_6$ we construct

$$\mathbf{e}_1\mathbf{e}_6\mathbf{e}_3, \quad \mathbf{e}_1\mathbf{e}_2\mathbf{e}_4, \quad \mathbf{e}_1\mathbf{e}_3\mathbf{e}_5, \quad \mathbf{e}_1\mathbf{e}_4\mathbf{e}_6, \quad \mathbf{e}_1\mathbf{e}_5\mathbf{e}_2.$$

Continuing out of $A_1 + \mathbf{e}_4$ is the cube $\mathbf{e}_2\mathbf{e}_3\mathbf{e}_5$; from $A_1 + \mathbf{e}_5$ is $\mathbf{e}_2\mathbf{e}_4\mathbf{e}_6$; from $A_1 + e_2$ is $\mathbf{e}_3\mathbf{e}_4\mathbf{e}_5$ and $\mathbf{e}_3\mathbf{e}_5\mathbf{e}_6$. Going still farther out from $A_1 + \mathbf{e}_2 + \mathbf{e}_5$ gives us the cube $\mathbf{e}_3\mathbf{e}_4\mathbf{e}_6$, and from $A_1 + \mathbf{e}_4 + \mathbf{e}_5$ comes the cube $\mathbf{e}_2\mathbf{e}_3\mathbf{e}_6$. Finally, from $A_1 + \mathbf{e}_2 + \mathbf{e}_3$ we construct $\mathbf{e}_4\mathbf{e}_5\mathbf{e}_6$.

When we write a sum of a point followed by a vector, we mean that the head of the given vector passes through the vectors in the sum. For example, the symbol $A_1 + \mathbf{e}_2 + \mathbf{e}_6$ says that the point A_1 has to pass through a segment having the length and direction of $\mathbf{e}_2$ and then going on through a segment having the length and direction of $\mathbf{e}_6$.

[Kowalewski's essay ended in a surprisingly abrupt way. It seems that the last section was a bridge into mathematical technicalities that he did not want to cross in an essay for general readers.

The *Supplementary Observations* that follow were originally at the end of Chapter Two. – Ed.]

Supplementary Geometric Observations

We still have to supply proof that the diagonals of a Kepler rhombus stand in a golden section ratio to each other. In Figure 16, two neighboring triangles of an icosahedron are shown. *ACB* is half of a Kepler rhombus based on *AB*.

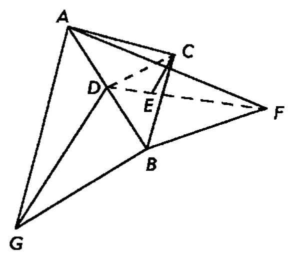


Figure 49

The rhombus forms an angle α with both triangles so the complete angle *GDF* is 2α, leaving a supplement of 2β needed to make two right angles. That is $\alpha + \beta = \pi/2$. When *CE* is the perpendicular line from *C* to *ABF*, then *E* will be the orthocenter, giving

$$DE = \frac{\alpha\sqrt{3}}{6} = \frac{\alpha}{\sqrt{3}},$$

where a is the side of the triangle. It follows from Figure 49 that

$$DC = \frac{DE}{\cos\alpha} = \frac{DE}{\sin\beta}.$$

The following relationship holds for the diagonal relation of the Kepler rhombus

$$\frac{DC}{DA} = \frac{1}{\sqrt{3}\sin\beta}.$$

From this we see that β is half of the angle between two neighboring planes of an icosahedron. To help in the calculation of β, one should now place a sphere of radius 1 about one corner of the icosahedron. The five planes of the icosahedron cut out a spherical pentagon whose sides equal $\pi/3$ and whose angles equal 2β (see Figure 50).

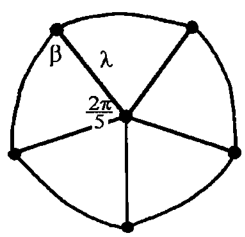


Figure 50

It can be read from Figure 50 that

$$\cos(\pi/3) = 1/2 \qquad \sin(\pi/3) = 1/2\sqrt{3}$$

$$1/2 = \cos^2\lambda + \sin^2\lambda\cos(2\pi/5)$$

$$\frac{\sin\beta}{\sin(2\pi/5)} = \frac{\sin\lambda}{\sqrt{3}/2}.$$

The first equation now gives

$$\sin^2\lambda = \frac{1}{2(1-\cos(2\pi/5))}.$$

From the second we obtain

$$\sin^2\beta = \frac{2\sin^2(2\pi/5)}{3(1-\cos(2\pi/5))} = 2/3\ (1+\cos(2\pi/5)) = 4/3\ \cos2(\pi/5),$$

thus

$$\sin\beta = \frac{2}{\sqrt{3}}\cos(\pi/5).$$

From this it follows that $\frac{DC}{DA} = \frac{1}{2\cos(\pi/5)}.$

In Figure 51, $\cos(\pi/5)$ is marked with the long bracket. We can see that $\cos(\pi/5) = ½(1 + z)$. In view of the fact that $z\,(1 + z) = 1$, we obtain $2\cos(\pi/5) = 1/z$.

$$\frac{DC}{DA} = z = ½(\sqrt{5} - 1).$$

With that we have obtained the desired result. It would have been possible to do it without the help of spherical trigonometry. We have supposed that most readers are familiar with the law of spherical sines and cosines. If that is not the case I recommend that you look it up in my book *Lehrbuch der hoheren Mathematik*, published by Walter de Gruyter, 1933.

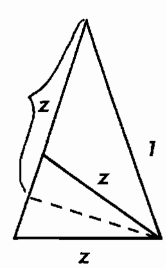


Figure 51

The famous psychologist, Fechner, founder of experimental psychology, concerned himself with the aesthetic properties of the golden section, and he has remarked that the most attractive rectangle has its sides approximately in a golden section ratio: They are nearly $\frac{1}{2}(\sqrt{5}-1)$ to 1. If we separate a square off such a rectangle, then a rectangle remains that is similar to the original. If we regard the long side of the rectangle to be of unit measure, then the short side is z. After removing the square, the remaining rectangle has sides of $1-z$ and z, and indeed $1 : z :: z : 1-z$.

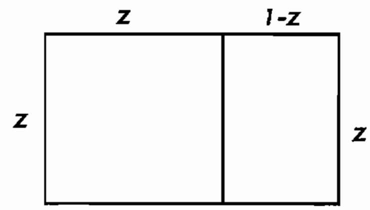


Figure 52

Figure 53 shows another remarkable property of the golden rectangle, that is a rectangle whose sides are in the ratio $z : 1$. Drop perpendicular lines from two oppositely positioned corners onto the diagonal that connects the other pair of opposite corners.

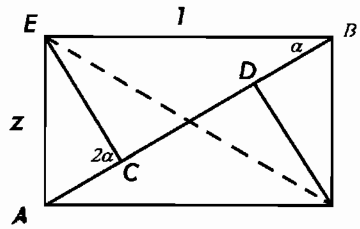


Figure 53

These two perpendicular lines are just as long as the diagonal segment between the feet of the perpendiculars.

So we get that $AC{\cdot}AB = z^2$, $CB{\cdot}AB = 1$. Making use of the fact that $AB = \sqrt{1+z^2}$, we obtain

$$AC = \frac{z^2}{\sqrt{1+z^2}} \qquad CB = \frac{1}{\sqrt{1+z^2}}.$$

This gives

$$CD = AD - AC = CB - AC = \frac{1-z^2}{\sqrt{1+z^2}}.$$

Furthermore $EC^2 = AC{\cdot}BC$, thus

$$EC = \frac{z}{\sqrt{1+z^2}}.$$

It follows from $1 - z^2 = z$ that $EC = CD$. One might also appeal to the fact, manifest in Figure 53, that $\tan\alpha = z$ and that

$$\tan 2\alpha = \frac{EC}{\frac{1}{2}C}.$$

Now make use of the fact that $\tan 2\alpha = 2$ to conclude that

$$EC = CD.$$

The golden section appears in the regular pentagon as shown in Figure 54.

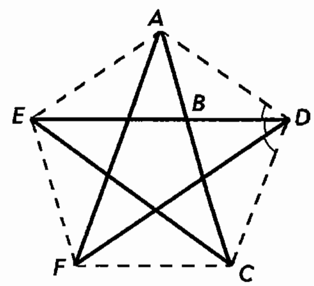


Figure 54

The three angles marked with a stroke are all equal to $\pi/5$; using the theorem on inscribed angles we obtain

$$ABD : BCD = AB : BC.$$

On the other hand

$$ABD{:}BCD = AD{\cdot}BD\sin(\pi/5) : BD{\cdot}CD\sin(2\pi/5) = 1{:}2\cos(\pi/5) = z : 1,$$

from which it follows that $AB : BC = z : 1$. This means that the diagonal AC is cut in a golden section ratio by the diagonal DE, and also naturally by DF. This gives that $AD : EC = AB : BC$, so we have that $AD : EC = z : 1$. The sides and the diagonals of regular pentagons stand in a golden section ratio. From this it follows incidentally that the angle between the diagonals of a pentagon, like the angle AFD of the figure, is equal to $\pi/5$.

Appendix
Rules and Tools David Booth

MacMahon's Blocks

Gerhard Kowalewski was the author of textbooks, treatises on analysis, and a mathematical autobiography [6]. It may have been his brother's study of *systematic color theory* that stimulated his interest in combinatorial geometry; it was not his usual field of research.

He began *Construction Games with Kepler's Solid* [5] with the puzzle of MacMahon's blocks. Here is the puzzle. One of the thirty blocks is chosen and set aside as a master. The thirty blocks are colored in the thirty different ways of putting six different colors on the faces of a cube. The master block could be any one of the thirty. The remaining blocks are used to make a two-on-a-side assembly whose outside colors agree with those of the master block.

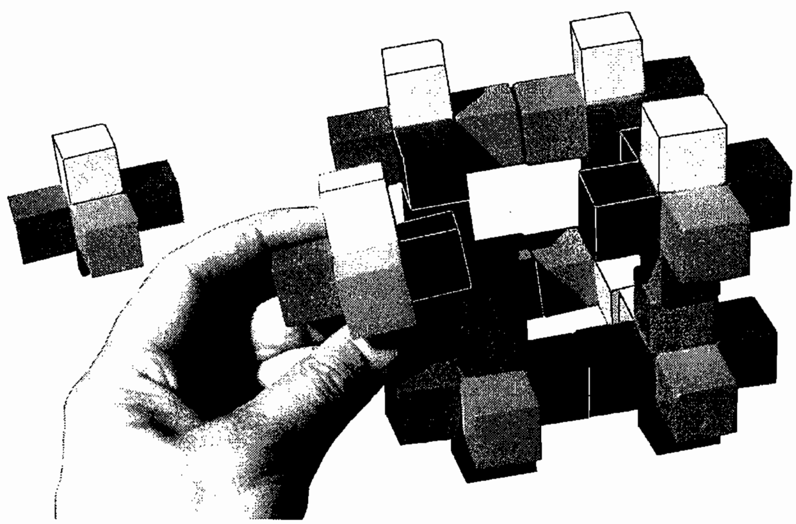


Figure 54

Figure 54 shows Kowalewski's version of the MacMahon blocks. Kowalewski cut away the blocks into cross shapes so that you can see into the assembly better. Instead of six faces on a cube, you see six little colored cubes attached to a central core.

At the left side in the picture is the master. This master block has its yellow side pointing up. So the top four blocks of the assembly must have yellow pointing up too. In Figure 54 the last piece of the puzzle is being put into place. The six sides of the two-by-two cube must have the same colors as the master cube.

The assembly is also required to obey the *domino principle.* When two blocks meet, they must meet on identical colors. In today's revival of tiling theory this would be called a *matching rule*.

Figure 55a shows the top view of the assembly in Figure 54. Yellow sides face up as dictated by the master block. On the bottom layer of the assembly the yellow sides cannot be exposed to the outside. They have to point inward to the center and they must meet other yellow sides in order to satisfy the domino principle.

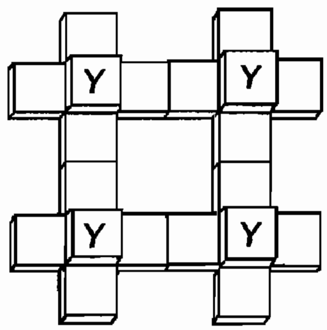


Figure 55a (top)

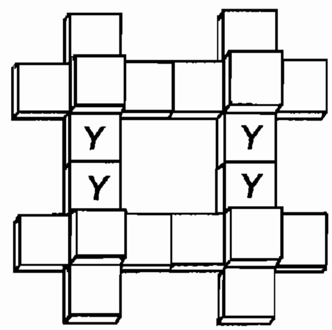


Figure 55b (bottom)

In actual play your hands and eyes work together to sort through the blocks, put aside the ones that cannot possibly be used, rotate the others into the proper position, and start the assembly. Someone who does it quickly would have good spatial intuition, dexterity, and practical intelligence. It is not necessary to ponder your moves. The first thing of interest is the question of whether a solution exists. Actually there are always two solutions.

The puzzle was described in the following way in P. A. MacMahon's book *New Mathematical Pastimes* in 1921, [7], a collection of tiling puzzles that were intended to illustrate combinatorial principles. We learn there that MacMahon, who wrote treatises on combinatorial mathematics as a retired British army officer, was shown the puzzle by his friend Col. Jocelyn.

> A cube has six faces, twelve edges and eight summits. If we are allowed six different colors in order to color the faces each with a different color, we find that we can make 30 differently colored cubes.
>
> It is a well-known rule, applicable to any regular solid, that in order to ascertain the number of different cubes or other solids that can be made by coloring the faces with different colors it is merely necessary to divide the factorial of the number of faces by twice the number of edges. Thus in the case of the cube we have
>
> $$\frac{6 \times 5 \times 4 \times 3 \times 2 \times 1}{2 \times 12}$$
>
> So also in the case of the tetrahedron, composed of four equilateral triangles, which has four faces and six edges we have
>
> $$\frac{4 \times 3 \times 2 \times 1}{2 \times 6}$$

And so on for any regular solid.

We now construct these 30 cubes and, denoting the colors by numbers, we represent any such cube in a diagram as shown in the figure....

They [the cubes] are conveniently denoted by 15 capital letters and by the same letters dashed, because they naturally arrange themselves into 15 pairs of cubes.

For example the cubes G , G' have the same pairs of opposite faces; the faces colored 1, 2, 3 are opposite to those colored 5, 4, 6 respectively in both cubes. If the colors upon any one pair of opposite faces of one of the cubes be interchanged the other cube is produced.

Looking vertically down upon the cubes the colors read clockwise on the one are identical with the colors read counterclockwise on the other.

The above cubes are called "associated cubes."

It is not obvious or even very easy to construct a Pastime from these 30 cubes. They can be assembled into a block having the dimensions $2 \times 3 \times 5$ and we can make a selection from the whole number in many ways; for instance if we can select intelligently either 8 or 27 of these they can be assembled into large cubes. Moreover we have four different contact systems at our disposal which, following the practice of other pastimes, we might denote by $C_{1,1,1,1,1,1}$, $C_{1,1,1,1,2}$, $C_{1,1,2,2}$, $C_{2,2,2}$.

The rule $C_{1,1,1,1,1,1}$ in MacMahon's notation is the ordinary domino rule: Colors can only meet like colors. The various other rules require crossed matchings. The $C_{1,1,1,1,2}$ matching rule, for example, has four colors that must meet a like color and one pair of colors that can only meet the other member of the pair. MacMahon continues.

It is now some years since Colonel Julian R. Jocelyn communicated to the present writer the fact that he could select eight cubes and assemble them on the contact system $C_{1,1,1,1,1,1}$ so as to produce a cube of twice the linear dimensions which is a faithful copy of the colors of any given member of the set of thirty cubes.

Suppose that it is desired to thus produce the cube denoted by A.

The two cubes A and A' have the same opposite pairs. Reject from the complete set all the cubes which have any pair of the opposites and it will be found that we are left with 16. These may further be divided into two sets of eight.

One of these sets of eight cubes can be assembled to give the two different solutions to Col. Jocelyn's puzzle.

The eight cubes in either case involve 48 faces and of these exactly half, viz. 24, are boundary faces. The remaining 24 are inside faces. It is a remarkable circumstance that the 24 boundary faces in the first solution are inside faces in the second and vice versa.

The geometry of the solutions can be further studied by taking advantage of the fact that the six centers of the six faces of a cube are the summits of a regular octahedron. The geometrical reader may be interested in following up this point.

MacMahon concluded his account of the puzzle by describing an interesting dual relation among the blocks. Say that "*A* supports *B*" if block *A* is one of the eight blocks used to make *B*. Then whenever *A* supports *B*, *B* supports *A* too.

Unfolding

Kowalewski's idea for extending the cube puzzle was to consider cubes in higher dimensions. Right away this suggestion offers many different higher dimensional analogies to familiar, everyday tile patterns.

Here is an example. Figure 56 shows the six square tiles each with edges that are colored with four colors without repetitions. They have been arranged in two ways to tile a cube while satisfying the domino rule. In this example, square, two-dimensional tiles are used to cover a three dimensional body, a cube. Furthermore, the covering is *exact*. By *exact* I mean that there are exactly six such 4-color tiles, and all of them are used.

This assembly, which was not actually mentioned by Kowalewski, is a natural one to consider when you think of projections that involve objects of different dimensions.

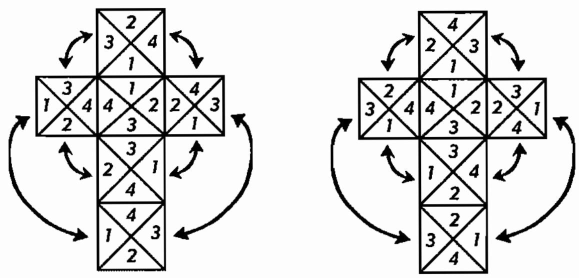


Figure 56

To analyze the tile pattern choose an arbitrary tile, the one with *1*, *2* , *3*, *4* in clockwise order will do, and turn the cube so that it faces you. You can now rotate the cube again so that color number *1* is at the top. There is one color number *1* on each tile, so there are three number *1* edges on the entire cube where the number *1* edges of tiles meet. These number *1* edges cannot be on the same tile because no tile uses the same color twice. Therefore two number *1* edges can be neither concurrent nor parallel edges of the same face. They must either be opposite parallel edges of the cube or skew. If they were opposite, however, the third number *1* edge would have to be concurrent or parallel to one of them across a face. So the number *1* edges can only be skew perpendicular edges of the cube.

Having turned the cube so that a number *1* edge is on top of the nearest face, there are only two distinct ways to find positions for the other two number *1* edges. One of the number *1* edges must point in the vertical direction. There are two choices; once the choice is made the position of the third skew edge is fixed.

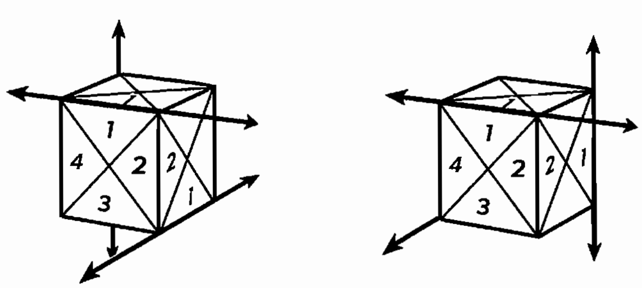


Figure 57

Figure 57 shows the two possible positions of the skew edges.

Knowing the position of these skew axes, color *1* can be painted astride all three of them. It remains to be seen how the rest of the cube is colored. Let x be an unknown color on the right-hand face.

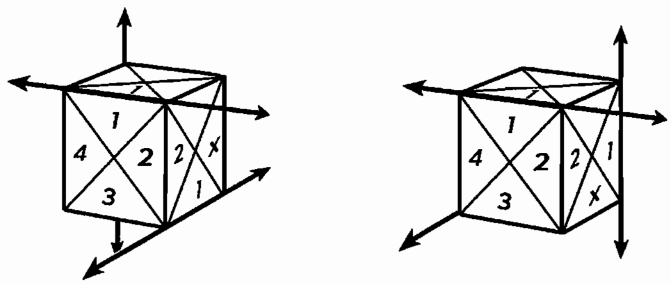


Figure 58

In the left-hand diagram of Figure 58, $x \neq 4$, because $x = 4$ would give two parallel edges of the same color, which can no more happen to color *4* than it can with color *1*. In the right-hand diagram $x \neq 3$, because $x = 3$ would give concurrent edges along the bottom of the cube. This observation fixes the colors on the right hand face in either case.

Figure 59 shows the result after solving for x in Figure 58.

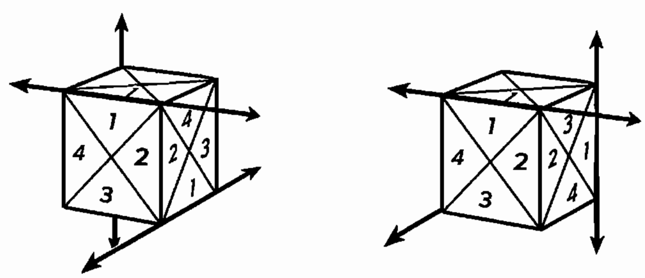


Figure 59

Two of the edges for color *3* are visible, so to continue you look for a position for the third number *3* edge. In the cube on the left we can immediately locate edge *3* on the rear face; on the right hand cube we can immediately locate edge number *3* on top. To find the remaining regions that are painted with color number *3* you look for a third edge that is skew perpendicular to the ones that have been found already. This technique can be continued to give a complete coloring to each of Figure 56.

Zonohedra

To investigate higher dimensional cubes in a lower dimension you can make use of polyhedra in three dimensions called *zonohedra*. When you project from a higher dimension, the projection generally distorts the angles.

The right angles of hypercubes, four dimensional analogs of ordinary cubes, are no longer right angles in the projected image, but their fundamental cubic structure remains as long as you use a parallel projection.

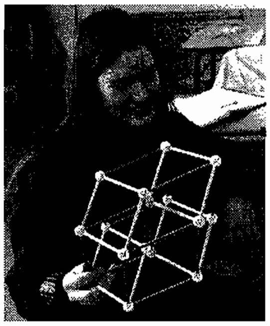


Figure 60

Figure 60 shows a general parallel projection of a four-dimensional cube into space, as does Figure 30 on page 38. There was a great interest in the fourth dimension in the early part of the twentieth century that was partly fueled by relativity theory, for more on this topic see [7]. Kowalewski had the idea of making games, they are more like puzzles really, that would be played on flat game boards. If you understood that the game board was a projection of a higher dimensional object you could solve the puzzle easily, but it would be difficult otherwise.

Projections of higher dimensional cube are known as *zonohedra*. Kowalewski used these zonohedra because the projections look like complicated designs, but in higher dimensions they are simple, cube-like structures.

Models of zonohedra and other kinds of projection from higher dimensional spaces were exhibited at scientific conferences in the United States by the Hartford, Connecticut oriental rug retailer, Paul Donchian, in the 1930s. Donchian's displays attracted news photographers to otherwise publicity shy mathematical conferences. When Albert Einstein, who was associated in the public view with the fourth dimension, visited the Donchian exhibit at the Chicago Fair in 1936 he had to come after hours, according to a Chicago newspaper, because the public fascination with the fourth dimension was so great that the curious crowd might have crushed both Donchian's models and the famous physicist.

The simplest example of these parallel projections that give diagrams of zonohedra are the plane representation of a cube. Figure 61 shows a hexagon that comes from projecting a wire frame cube that has one corner pointing straight up.

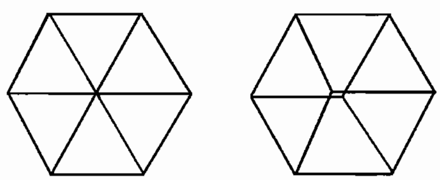


Figure 61

The diagram at the left looks like a flat hexagon divided into triangles. In order to make it look more like a cube it is best to tilt the axis of projection slightly so that the two opposite corners do not merge into one single point.

It is clear that the square faces on the original three-dimensional cube have become distorted into parallelograms. For his puzzles, however, all that Kowalewski needed was the proper relationship between corners, edges, and faces. It does no harm to have angles distorted in a projection from higher dimensional space.

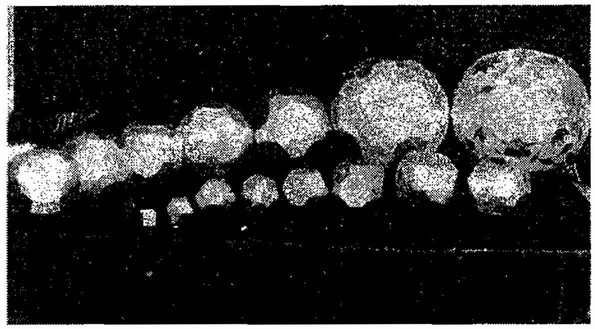
Paul S. Donchian Opens Door To His Wire And Cardboard Models

Professors Label Them Instructive

Declare Hartford Man's Inventions Help Them and Students Visualize Even 24 Dimensions

BY PHILIP BECKWITH

A contribution to science that had the wizards of higher mathematics perking up their ears, the photographers hot-footing from newspaper offices, and drew a gaping public to witness four or five dimensions has recently been returned to Hartford from Pittsburg Pa., where it had been on display. Few except the experts attending the Pittsburgh meeting of the American Association for the Advancement of Science knew much



Mr. Donchain's novel march of the dimensions showing progression from a small three-dimensional cube in the left foreground to the largest figure at the right, a projection on the space of perception of a 24-dimen -

New Tricks in Geometrics

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Figure 62

The most symmetrical zonohedra are the rhombic dodecahedron, which can be treated as the projection of a four-dimensional cube, and the rhombic triacontahedron – or Kepler's Solid – a projection of a six dimensional cube.

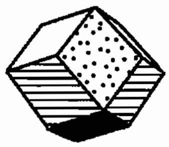

Figure 63

The rhombic dodecahedron is shown in Figure 63. Kowalewski did not begin right away by introducing this solid figure as a projection from four dimensions. Instead he described, it in a down-to-earth way as the shape that you get if you put skirts on a cube.

To put skirts on a cube you place peaks on each of the cube's faces in such a way that the slopes of neighboring peaks are in alignment. Figure 64 shows two such mountains that are aligned in a flat slope.

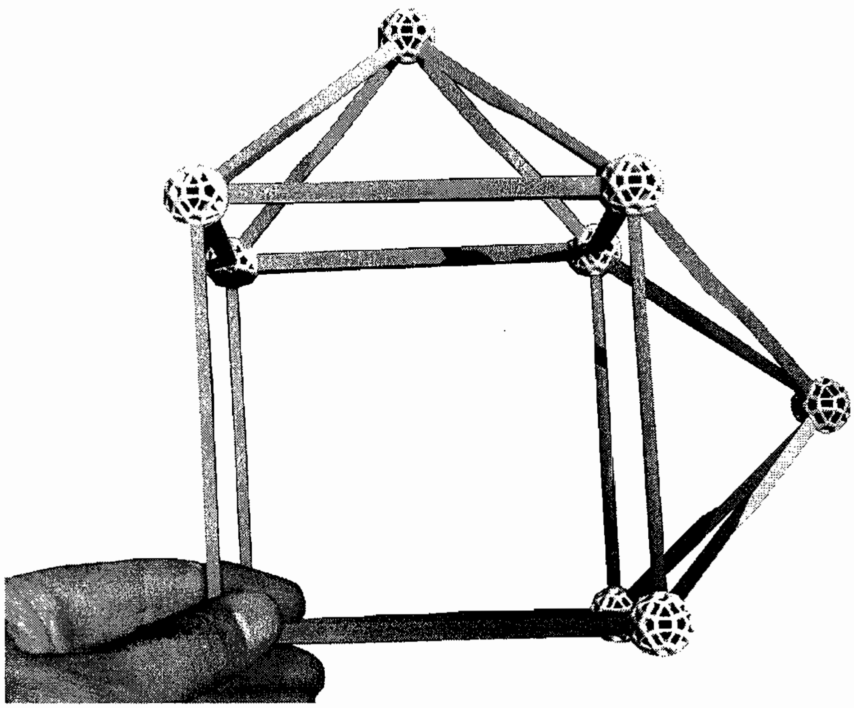

Figure 64

When you have put on all the skirts, so that every face of the original cube has an attached peak, the outside of your model will be made up of rhombi. These *Moraldi rhombi* have diagonals that are in the ration 1: $\sqrt{2}$.

Removing the interior cube leaves a rhombic dodecahedron (*dodecahedron* because there are *twelve* rhombi on the exterior surface.)

It is not quite right to say that a rhombic dodecahedron is the projection of a four-dimensional cube, because the hypercube has 16 corners and the rhombic dodecahedron has only 14. There are two corners in the center of the rhombic dodecahedron just as there are two opposite cube corners in the center of the hexagonal projection of a cube in Figure 61. A zonohedron is actually the external shell of the projection of a higher dimensional cube.

The four axes from four-dimensional space are arranged symmetrically in the projection that creates a rhombic dodecahedron. In fact the rhombic dodecahedron is so symmetric that it is easy to lose sight of its origin in four-dimensional space. The number four, however, is preserved in the zones of the rhombic dodecahedron. Each zone consists of a band of parallel edges that form a belt around the body.

It is interesting to make a cardboard model that shows the bands in four different colors weaving alternately over and under each other.

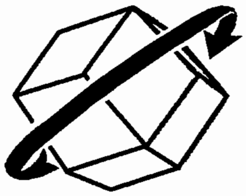

Figure 65

Figure 66 shows a net that can drawn on a larger scale, scored along the edges, folded and glued to make a cardboard model with the four zonal bands showing. Choose four different colors for the bands. If you make one, you will need to leave tabs on the edges for gluing. It is easier to make than the corresponding model of Kepler's Solid .

How can we make a settlement puzzle out of our rhombic dodecahedron? Three possibilities are mentioned by Kowalewski. You could fill the solid figure with three-dimensional blocks, you could successively cover the rhombic faces or you could occupy the 14 corners of the zonohedron.

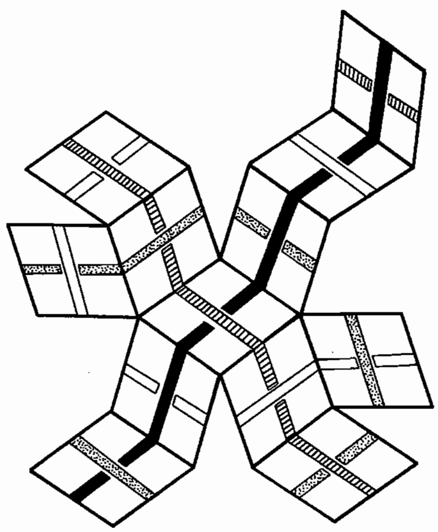

Figure 66

Kowalewski had the urge to make games out of these puzzles, but none of his ideas were effective games, because the players take turns working on a settlement puzzle. One player's moves have little effect on the choices of another.

A settlement puzzle could involve points, surfaces, or cells. The most natural settlement of surfaces is to lay rhombic tiles colored to show the zonal bands like the model in Figure 66. Kowalewski did not describe such a game; he had already discussed the game of thirty little men, a similar game played on the faces of the rhombic triacontahedron, Kepler's Solid.

What about settling the *cells* of the rhombic dodecahedron? You cannot use all eight cells that cover the surface of a hypercube in four dimensions because they overlap each other in the projection that creates the rhombic dodecahedron. It is possible to settle the rhombic dodecahedron, however, with four of the blocks without overlapping. Kowalewski thought about coloring the edges of the constituent blocks in four different colors. You place them in the rhombic dodecahedron with the like colors pointing in the same direction so that they are aligned with the zones of the enclosing form.

You could also try packing blocks that have opposite faces colored alike, though Kowalewski does not mention this settlement. Figure 67 shows how it can be done.

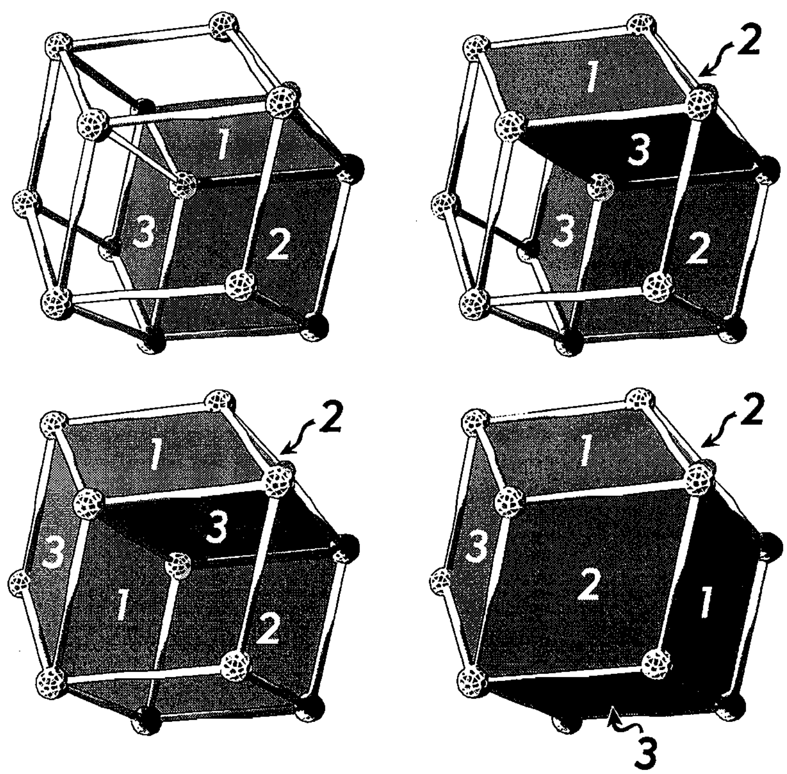

Figure 67

There is only one way to color a single block so that opposite faces agree. If the colors are in the order *1*, *2*, *3* when you look at one obtuse corner, they will be in the reverse order on the back side. The four identical blocks can easily be fit into the rhombic dodecahedron.

Having tried settlement puzzles on the faces and cells of a rhombic dodecahedron, there are only the corners left among its geometrical elements.

A corner can be in contact with as many as four other corners. The game board marker that we use must have four different properties to represent contact with its neighbors. Kowalewski describes them as towers in Chapter Five. Each tower has four stripes that could be either of two colors.

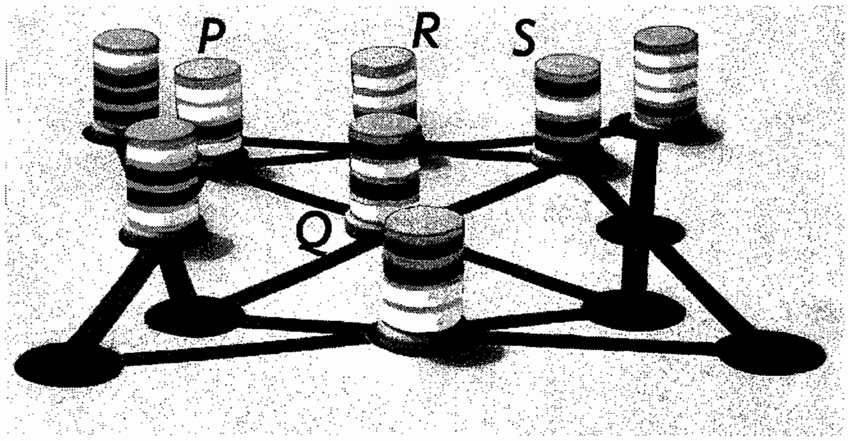

Figure 68

Figure 68 shows a game board on which some of the towers have been properly placed. Towers that are joined on an edge of the game board have three bands in agreement and the fourth one different. For example, *P* and *Q* differ only on their top bands.

The trick in settling all the towers is to focus your attention on the four-sided regions that make up the game board. Opposite edges of one of these regions are parallel to each other back in four-dimensions where we started mathematically. These opposite edges are part of the same zone in three dimensions. The projection onto the game board is made from a one-point perspective that destroys the parallelism. The geometrically knowledgeable player, however, knows that opposite edges of a quadrilateral region originate as parallel lines and therefore the same band of a tower has to change along this edge as along its opposite. The towers *R* and *S* are placed across a quadrilateral from *PQ*, so they must also differ only at their top stripes. If a person does not know this, he will play a few towers and then get stuck; but a player who knows the winning strategy can play all the towers. As soon as someone realizes there is a strategy they lose interest, Kowalewski claims, because "The purest pleasure can only be obtained by a childish soul who is unburdened by the theory."

Use of the Zometool

Kowalewski's essay [5] and H. S. M. Coxeter's book *Regular Polytopes* [3] helped stimulate the design of the *Zometool* (additional information can be found in Baer [2], so it is not at all surprising that this construction system greatly facilitates the construction of models described by Kowalewski. Few us of today have the calm leisure needed to undertake model making as a hobby. The *Zometool* allows you to see the geometrical relationships quickly. You can decide later whether you want to make a cardboard model for display.

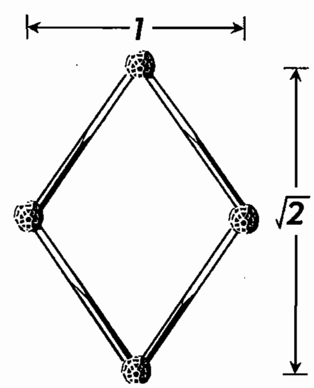

Figure 69

When you use the *Zometool* to make a rhombic dodecahedron (Figure 64) by putting skirts on a cube the inner cube is made with blue struts and the mountains are yellow ones. The rhombic faces are shown in Figure 69.

It is even more important to have some *Zometools* at hand when you go on to work with Kepler's Solid, the rhombic triacontahedron. Construction of the cardboard models recommended by Kowalewski are more difficult and time consuming than making a rhombic dodecahedron. A *Zometool* model can be made very quickly and it reveals the geometrical relationships well.

Kepler's Solid

This brings us to the heart of Kowalewski's essay the properties of Kepler's Solid, a zonohedron of six zones.

In the first place Kepler's solid can be made by putting skirts on regular dodecahedron. A vast amount of labor is saved by having some *Zometools*. First make a regular dodecahedron. If you have never made one before, it is best to use the porcupine method.

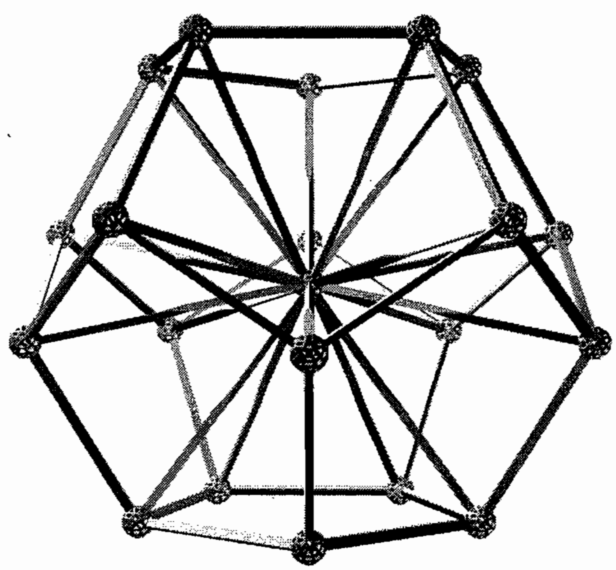

Figure 70

Fill a central *Zometoo*l ball with as many long yellow struts as it will hold. When you have that "porcupine" then connect the ends with middle length blue struts. The blue struts make a regular dodecahedron. The yellow lines of the porcupine are just there to help in the construction. After you have done this once, you will see how to make a regular dodecahedron without the help of the yellow porcupine.

You can put skirts on the regular dodecahedron just as with the cube. The resulting surface of 30 rhombi is the rhombic triacontahedron, what Kowalewski calls "Kepler's Solid." The rhombi meet five at a corner at their acute angles and three at a corner at their obtuse angles.

These rhombi that form the outside shell have diagonals that are in the proportion 1: τ, where τ is the number of the golden section $\frac{1}{2}(1 + \sqrt{5})$. They are made with red struts in the *Zometool* system.

The blue lines of the regular dodecahedron that was obtained on the porcupine are all short diagonals of the golden section rhombi. When you have Kepler's solid, you can remove the blue lines.

It is actually quite easy to make Kepler's Solid without the help of a dodecahedron to dress up in skirts. You need sixty red Zometool struts of the same length. Make rhombi with the red struts and assemble them so that each stick has a 5-way corner at one end and a 3-way corner at the other end. A model with solid faces is shown in Figure 71.

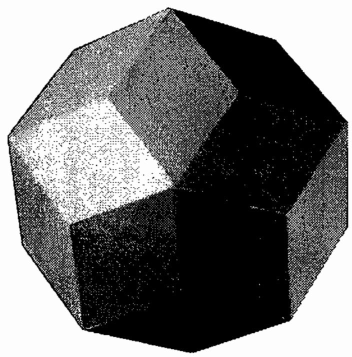

Figure 71

To fill the interior volume with parallelepiped blocks you need thirty blocks. They are of two kinds "steep" and "stubby". You can get the idea of how to fill it with *Zometools* struts. To really follow Kowalewski at this point, however, you need to have a cardboard model or panels to attach to *Zometool* struts.

When the blocks are assembled in Kowalewski's settlement puzzle five different colors are used to make an interesting model. If the model comes apart, it could take you quite a while to get it back together again, so Kowalewski gave directions that told how to reassemble it (pages 26-29).

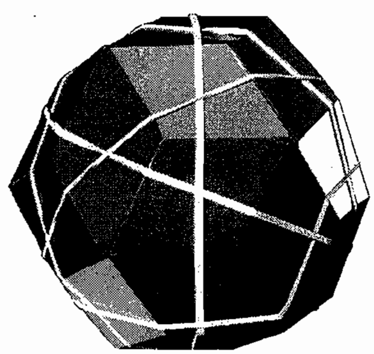

Figure 72

In addition to this puzzle involving blocks, there are possible puzzles using either the edges or corners of Kepler's Solid.

The puzzle that comes from settling the corners of the rhombic triacontahedron resembles the similar puzzle, based on the rhombic dodecahedron, that uses towers of four stripes. This time the towers would have to have six stripes because the rhombic triacontahedron has six zones. Kowalewski did not mention this settlement, probably because it would be hard to interest anyone in it.

He did have enthusiasm for the settlement puzzle on the faces of the rhombic triacontahedron. He imagined this puzzle as one in which little men were used as game markers. Of course the actual play is done on a projection of the solid body onto a game board net. This has the virtue of practical convenience as well as that of disguising the origin of the game from the naïve player. The knowledgeable player knows that opposite edges of a quadrilateral on the game board belong to the same zone of the solid body. You play so that either the pants or jacket of a man matches that of any neighbor. The zonal bands can be thought of as going alternately over and under around the triacontahedron. The strategy tells you to play your men in that way too. As you follow a zone around the board, say the zone corresponding to the color red, the men will have red pants, red jackets, red pants, red jacket, and so on.

There has been a steadily increasing mathematical interest in settlement problems as a branch of geometry. Gründbaum and Shepard's *Tilings and Patterns*, [4], gives an extensive account of the two-dimensional case. It can be confidently predicted that the beauty of the subject will ensure continued interest. It is probably only a matter of time until there is a revival of interest in the architectural applications, [1].

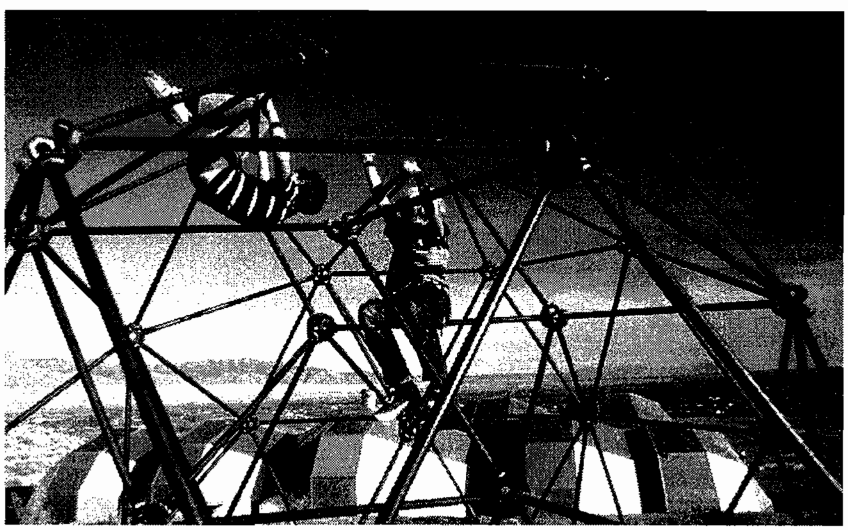

Figure 73

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Index of Puzzles

9. Parallelogram Tiles Cover a Rhombic Triacontahedron.

Chapter Four, pp. 31 - 35.

This is the puzzle of the thirty little men with variously colored pants and jackets. They are equivalent to the use of 30 parallelogram tiles in six colors. Each tile can use only two colors because parallel edges are colored alike.

10. Markers on the Corners of a Rhombic Triacontahedron.

Chapter Six, pp. 59-77.

Kowalewski did not discuss this as a puzzle. To treat it in the same way as Puzzle 7 would require towers with six bands. He did treat it geometrically.

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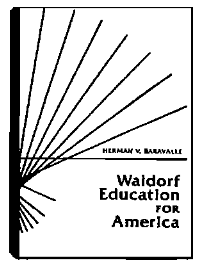

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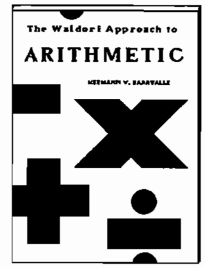

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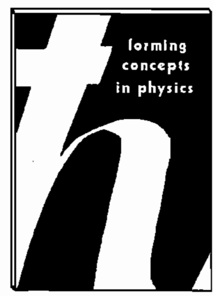

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