Zome Geometry

Hands-on Learning with Zome™ Models

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Truly I begin to understand that although logic is an excellent instrument to govern our reasoning, it does not compare with the sharpness of geometry in awakening the mind to discovery.

—Galileo
(said by Simplicio in *Dialogs Concerning Two New Sciences*)

Introduction

Welcome to *Zome Geometry*! We have found the Zome System to be a wonderful tool. With it, we have deepened our understanding of many geometric ideas in two, three, and even four dimensions—especially, but not exclusively, ideas about polyhedra. In this book, we share our favorite activities with students, teachers, and, in fact, any interested reader.

How to Use *Zome Geometry*

This book can be used in several ways.

- *Zome Geometry* activities supplement the secondary curriculum. The prerequisites and the specific curricular connections (from geometry, trigonometry, algebra, and more) are listed in the teacher notes at the beginning of each unit. Some activities preview or introduce the corresponding topics; others are more suited for the review or application of previously studied topics; and many will work both ways.
- *Zome Geometry* can serve as the textbook for a mathematics elective course.
- You can use *Zome Geometry* as a source of projects for math teams, math clubs, or individual students.
- Anyone who wants to explore geometry on his or her own can use *Zome Geometry* as a self-instruction manual. Read the answer only after working on a question, and make use of the teacher notes as you work through the activities.

The time that students invest in laboratory-style activities deepens their understanding and increases their motivation to study geometry. As with laboratory exercises in a science course, the *Zome Geometry* activities provide directions and insightful questions but leave key observations and discoveries to the explorer.

How *Zome Geometry* Is Organized

Each unit focuses on a specific content area and is divided into lesson-size activities. Each activity starts with a Challenge, which students can explore on their own. Preferably, they will do this without access to the rest of the activity, which often answers the Challenge. The Challenge is intended to lay the groundwork by posing a question that will be developed in the guided activity. In some cases, the Challenge is very difficult, and you will need to decide how much time you want to allot for it. Each unit ends with Explorations, which expand or deepen what students have learned in the core of the unit.

If students have built a structure, do not assume that they understand everything about it. While the hands-on work is very helpful in seeing relationships, students may have trouble focusing on concepts while they are building. There are two types of exercises in the core activities—building prompts (1, 2, . . .) and Questions (Q1, Q2, . . .). The Questions give students a chance to reflect, and they increase the likelihood that students will make interesting discoveries while building. Depending on the availability of time and materials, students may do the building exercises in the order given, or they may do a certain amount of building, and then go back to answer Questions. In any case, for students to take full advantage of *Zome Geometry*—and to enable what they learn to transfer to other parts of their mathematical world—it is essential that they come up with written, thoughtful answers to the Questions. Their answers may follow or generate small-group or whole-class discussions.

In general, it is best to keep models intact as long as possible and to keep them within reach of the students. If models need to be stored overnight and shelf space is limited, they can be hung on paper-clip hooks. If you have easy access to a camera, taking pictures of models can be a useful way to document progress and to create records that can be referred to when working on later activities. In some cases, taking photos of half-completed structures yields images that are easier to decipher than pictures of completed projects.

The Explorations that close each unit, and the additional unit of Explorations that closes the book, tend to require more time and materials than the other projects. Their mathematical level is often a bit higher than that of the core part of the activity. Hence, they are well-suited for term papers, extra-credit projects, or challenges for the more ambitious students in a class.

About the Materials

The zomeball is designed with rectangular holes for blue struts, triangular holes for yellow struts, and pentagonal holes for red struts. Green and green-blue struts, which also fit into the pentagonal holes, have been added to the Zome System, so that Zome models of all the Platonic solids can be built. An illustrated list of the strut names is given on page 265. You may want to photocopy this page and display it for reference. Blue, yellow, red, green, and green-blue struts are all available in the special Creator Kit designed for this book and distributed by Key Curriculum Press. If you already have Zome materials, green struts can be purchased separately. In this book, we will refer to both the green and green-blue struts as *green*.

The green struts are challenging to use. Builders should have practice with the blue, yellow, and red struts before they face the challenge of distinguishing between the five different angles that the green struts can fit into a given zomeball pentagonal hole. Follow the instructions in the teacher notes for Unit 3 as you and your students learn to build with the green struts. Most of the activities can be done without the green struts, however. Many involving regular tetrahedra and octahedra made with green struts can be approximated with non-green tetrahedra and octahedra.

The special Creator Kit with green struts includes enough pieces to build all but the most complicated models and the big domes. This kit will be enough for two or three groups to work on many of the activities. However, if you want to build the more complex models or to divide the class into four or more groups, you will need two kits. The index of polyhedra on page 259 indicates the Zome materials required to build each model. Use this list to help you figure out if you have enough materials for a given activity. To conveniently distribute materials at the beginning of class, and to clean up at the end, you may organize your struts and balls in labeled resealable plastic bags. You should have as many bags for each type of strut as you have groups of student builders.

Many of the complex models will take some time to build. If several models are built during an activity, you can conserve Zome materials and time by having each group build a different model and then share the models as students answer the Questions. Also, it is often possible to conserve materials by having different groups of students build the same models in different sizes.

About Frequently Built Polyhedra

Some polyhedra are studied again and again throughout the book. Students will learn to build them quickly. If you embark on an activity out of sequence, however, students may encounter an unfamiliar polyhedron whose building instructions were in a skipped activity. In such cases, consult the index of polyhedra, which lists in bold the activity or activities that contain the building recipe for each polyhedron.

About Numerical Answers

When appropriate, require that students give both calculator-generated numerical answers with reasonable accuracy, such as 1.618, and mathematically exact answers, such as $\frac{1+\sqrt{5}}{2}$. The former are important if students need to know the magnitude of a number, as for comparison purposes. The latter are important because, in many cases, they facilitate communication and deepen understanding of mathematical relationships.

About Proof

We strongly believe that students should be introduced to formal mathematical proof, but that is not the only thing that needs to happen in math class. Because this book will be used as a supplement to a wide range of math classes, with students at many different ages and levels of mathematical maturity, we have not emphasized formal proof.

Instead, the main purpose of *Zome Geometry* is to introduce students to a beautiful part of geometry, to reinforce their visual sense and spatial intuition, to give them a chance to apply ideas they have learned in other math classes, and to make interesting connections between different areas of mathematics.

We do consistently ask students to reason about the figures they build—a necessary prerequisite to formal proof. Moreover, a few important activities lead students through logically tight arguments about the mathematical properties of the Zome System (see Units 7 and 13) and about three fundamental theorems concerning polyhedra (see Activities 3.3, 24.1, and 24.2).

About the Authors

George W. Hart (http://www.georgehart.com) is an artist and mathematician. Henri Picciotto (http://www.picciotto.org/math-ed) is a mathematics teacher at the Urban School of San Francisco and a curriculum developer specializing in hands-on materials.

A list of updates and corrections for *Zome Geometry* will be available at http://www.georgehart.com.

We hope that you not only learn from this book, but also have a very good time.

UNIT 1

Angles, Polygons, Polyhedra

Students are introduced to the Zome System and its notation. They become familiar with the zomeball and struts and learn which polygons can be built. Then they begin to build in three dimensions and learn how to scale Zome models to produce a similar polyhedron.

Goals

- To become familiar with the Zome System and the notation used to refer to the components
- To discover some of the angles in the Zome System
- To build regular polygons (3, 4, 5, 6, 10 sides)
- To explore the structure of prisms, antiprisms, and pyramids
- To investigate scaling Zome polyhedra

Prerequisites

Students need to know about the angles of a regular *n*-gon.

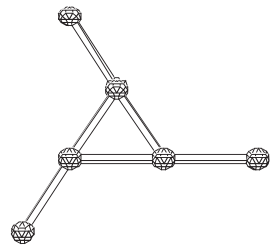
Notes

These models are small and can be made with the red, blue, and yellow struts from any Zome System set. Do not use the green struts until Unit 3.

If your students are not familiar with the angles of regular polygons, use the following discussion to introduce exterior angles.

> A regular *n*-gon has an exterior angle of $^{360}/_{n}$. This can be explained with the following argument. Imagine you are driving a car along the edges of a polygon. At each corner you make a sharp turn, which is the exterior angle at that vertex. For example, if you are driving around a regular hexagon, each turn is 60 degrees. (0 degrees means you don't turn, you just go straight.) When you return to your starting point and turn to face the same way as when you started, you have made one complete revolution, that is, 360 degrees of turns. Since there are n equal turns, each turn is $^{360}/_{n}$ degrees.

You may illustrate this argument by making a Zome equilateral triangle, with each side extended as in the figure. A regular pentagon or hexagon would work just as well, but avoid the square as a first example, as its interior and exterior angles are equal.



Triangle with sides extended

Make sure students realize that, at any vertex, the interior and exterior angles add up to 180 degrees.

1.1 Angles and Regular Polygons

Most of the work in this book involves regular polygons, so this introduction is essential. These activities also preview later work on Zome symmetry. Make sure students understand that a regular polygon has equal sides and equal angles and that the angle relationships in the starburst correspond to those in a regular polygon. Suggest that different students build the polygons in different sizes. Advise students to keep the polygons they make, as they will need them later in this unit.

The regular 8-gon requires green struts and is covered in Unit 3. An understanding of why other polygons such as the 7-gon, 9-gon, and 11-gon cannot be constructed with the Zome System follows from the tabulation of all Zome-constructible angles in Unit 13.

1.2 Prisms, Antiprisms, and Pyramids

The activity begins with informal definitions. You may want to mention that the Egyptian pyramids are regular 4-gonal (square) pyramids.

This is the first of many opportunities to count vertices, edges, and faces of polyhedra. As students determine these numbers, encourage them to use logic and think of the structure of a Zome model, rather than count items one by one.

1.3 Zome System Components, Notation, and Scaling

As students become familiar with the zomeball, be sure they notice that for two connected balls, struts inserted in corresponding holes in the two balls will be parallel. Although only certain directions are possible, the set of directions is the same on every ball.

1.1 Angles and Regular Polygons

Challenge

Determine which different regular polygons can be made with the blue, red, and yellow struts of the Zome System. Don't consider different sizes, just different shapes.

1. Take a zomeball and put it on the table so that it is resting on a pentagonal hole. It will also have a pentagonal hole facing straight up, since each hole is opposite another hole of the same shape. Think of the bottom and top pentagonal holes as south and north poles. Stick ten blue struts (any size) into the ten rectangular holes along the equator. They will all be horizontal, defining a flat "starburst" of ten equally spaced rays.

   Q1 With a pentagonal hole at the pole, what is the angle between consecutive equatorial struts?

2. Put a triangular hole at the pole, and make a starburst along the new equator.

   Q2 With a triangular hole at the pole, what is the angle between consecutive equatorial struts?

3. Put a rectangular hole at the pole, and make a blue starburst along the new equator.

   Q3 With a rectangular hole at the pole, what is the angle between consecutive equatorial struts?

   Q4 Describe the pattern in the relationship between the number of rays in each starburst and the shape of the hole at the pole.

   Q5 Using the angles you found in Questions 1, 2, and 3, you should be able to make regular n-gons for five different values of n. What are the values of n?

4. Build any kinds of regular polygons you didn't already build during the Challenge. Look for polygons with different shapes, angles, and numbers of sides. You don't have to build different sizes.

   Q6 Place the regular polygons on the table, and make a chart relating the shape of the hole at the zomeball pole with the corresponding n.

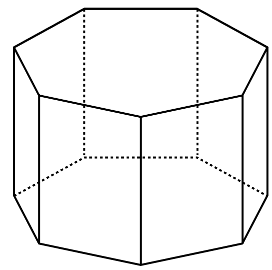
A more advanced study of zomeball angles shows that there are no other regular polygons constructible with the red, yellow, and blue struts. The regular 8-gon will be constructed using the green-blue struts.

1.2 Prisms, Antiprisms, and Pyramids

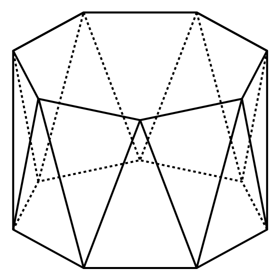
Challenge

Determine which right prisms, which antiprisms, and which pyramids can be built with the Zome System.

A three-dimensional analog to the polygon is a prism. A *right prism* is shaped something like a drum, but with an *n*-gon (instead of a circle) for top and bottom and rectangles around the sides. An *antiprism* is a fancier kind of drum-shaped polyhedron. It also has an *n*-gon as top and bottom, but they are rotated with respect to each other so that the vertices of the top one are between the vertices of the bottom one. The sides of an antiprism are triangles instead of rectangles.



7-gonal prism



7-gonal antiprism

An *n*-gonal pyramid has an *n*-gon for a base and *n* triangles for sides. The vertex where all the triangles meet is called the *apex*.

1. Make a regular pentagon for a base (use any size of blue struts). Then place a red strut (any size, but all five the same) into the north pole of each of the five zomeballs. Top each red strut off with another zomeball, and connect them to make a second regular pentagon. Your pentagonal prism is a three-dimensional solid bounded by two pentagons and five rectangles. The shape of the rectangle depends on which size struts you chose.
2. Make a prism using a triangle, a square, a hexagon, or a decagon as base. (The sides will be rectangles, and, in the case of the square base, they can be squares.) Make a prism different from your neighbors'.

Q1 Build a square prism with square sides. What is another name for this polyhedron?

A *vertex* is a corner of a polygon or polyhedron where the edges meet. It is represented by a zomeball. The plural of *vertex* is *vertices.*

Q2 For an *n*-gonal prism, write formulas that involve *n:* for the number of vertices, for the number of edges, and for the number of faces.

3. Make a pentagonal antiprism: Connect the top and bottom pentagons by a zigzag of edges, making ten equilateral triangles, half pointing up and half pointing down. (If you have trouble doing that, turn your base pentagon upside down.)

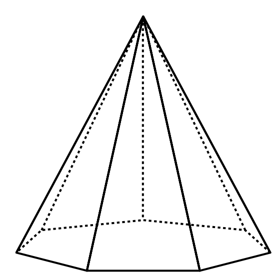
There are five different shapes of pentagonal antiprisms that you can make with the lengths and angles in the Zome System. They differ according to how far you raise the top pentagon and which size strut you use to connect the two pentagons.

4. With your neighbors, make the other four Zome pentagonal antiprisms. Look for different shapes and angles, not different sizes. The zigzag might be red, yellow, or blue. (Hint: One is very short. Remember that turning over your first pentagon may help.)
5. With your neighbors, build five different shapes of triangular antiprisms. The zigzag might be red, yellow, or blue.

Q3 For an n-gonal antiprism, write formulas that involve n: for the number of vertices, for the number of edges, and for the number of faces.

6. Make a Zome square pyramid.

Q4 How many different Zome pyramids can you make on an equilateral triangle base? (Hints: Try a base made with the medium-size blue strut. Once you have exhausted the possibilities with one side of the base up, turn the base over.)


7-gonal pyramid

Q5 How many different Zome pyramids can you make on a regular 5-gon base?

Q6 For an n-gonal pyramid, write formulas that involve n: for the number of vertices, for the number of edges, and for the number of faces.

1.3 Zome System Components, Notation, and Scaling

Challenge

Make antiprisms in four different sizes, but with the same angles.

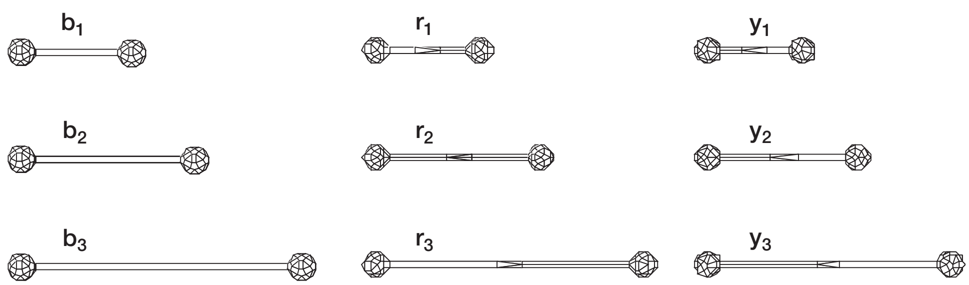
In making prisms, you relied on an important property that is designed into the Zome System: Whenever balls are connected, they have the same orientation. As a result, you can always construct a line parallel to any given strut from any other connected ball.

Q1 Hold up a zomeball and look into a rectangular hole, through the center of the ball, and out the opposite rectangular hole. Notice that the long sides of these two rectangles are parallel. Do the same with a triangular hole and a pentagonal hole. What do you notice?

Q2 Hold any blue strut vertically and examine it. Notice that, disregarding the ends that fit into the zomeball, it can be seen as a tall prism with a rectangular base. Do the same with any yellow strut and any red strut, and notice that each is composed of the connecting ends and three distinct structures in a stack. What are the three distinct structures of these struts?

Q3 Describe what the twists in the red and yellow struts have to do with the zomeball.

The struts are named as shown in the figure, with **b** for blue, **r** for red, and **y** for yellow. For each color, there are three sizes, which are numbered 1, 2, and 3.



Zome struts

Scaling a polyhedron can be done by adding 1 to each size of strut. For example, if you made an antiprism using b_2s for the pentagons and r_1s for the zigzag, the next larger size uses b_3s and r_2s in the corresponding places. When scaling a figure, be sure that angle measures are unchanged!

1. Pick one of the five pentagonal antiprisms you made, and make one similar to it, but scaled up or down one size.

Another approach to scaling is to double (or triple, and so on) the number of struts on each edge. For example, if you had edges of length b_2, connect two b_2 struts with a ball, and use length $2b_2$ for the corresponding edges in the scaled polyhedron.

Q4 Visualize the shape of a cross section of a pentagonal antiprism halfway up between the base and the top. What kind of polygon is it?

2. Build a model that explicitly shows this cross section. Pick any one of the five pentagonal antiprisms and build it in double scale. The balls at the halfway points of the zigzag struts are just where you need them to make the cross section.
3. Make a triangular antiprism in double scale to show its cross section halfway up.

Q5 In general, what polygon do you see if you slice an antiprism with a cut parallel to and halfway between its *n*-gons?

Q6 Explain why you cannot use the Zome System to make an antiprism with a 6-gon or 10-gon base. You may assume that 12-gons and 20-gons are not Zome-constructible.

Explorations 1

A. Skew Polygons A *regular skew polygon* is a polygon with equal sides and equal angles that does not lie in a plane. Instead, its vertices lie on an imaginary cylinder, in two parallel planes. Make a cube and let it hang down from a vertex; notice that around its "equator" is a regular skew hexagon. The zigzag of an antiprism is a regular skew polygon. How many different kinds of regular skew polygons can you build with the Zome System?

B. Nonright Prisms The prisms you made are *right prisms* because the edges connecting the two *n*-gons are at a right angle to the plane of the base. Find some Zome nonright prisms.

C. Another Cross Section What is the cross section of an antiprism one third of the way from one base to another?

D. Rhombic Pyramids A *rhombus* is a planar 4-gon with equal sides, but the angles do not need to be equal. The plural of *rhombus* is *rhombi.* Find four different kinds of blue rhombi, meaning that they have different angles from each other. A square is one of the four, since it is a special kind of rhombus. You should be able to determine the angle in each of your four blue rhombi. There are also one kind of red rhombus and two kinds of yellow rhombi. Build them.
A rhombic prism is easy to build; as more of a challenge, can you make a pyramid on each of your rhombi? (The apex need not be directly over the center of the rhombus.)

E. Concave Antiprisms The prisms and antiprisms above are all convex—they have no indentations. You can also make a symmetric concave antiprism. Start by making a 5-gonal antiprism with b_1 pentagons and b_1 slanting edges. (Its ten sides are equilateral.) Think of the zigzag as a cycle of ten struts numbered 1, 2, 3, . . . , 10, and remove the odd-numbered ones. That creates five openings, each like a rhombus, but not planar. Put in a b_2 as the long diagonal of each "rhombus." Describe the result. How is it like and unlike the other pentagonal antiprisms?

UNIT 2

Icosahedron and Dodecahedron

Students are introduced to the icosahedron and dodecahedron and use these regular polyhedra to explore the effects of scaling.

Goals

- To learn about the structure of the icosahedron and dodecahedron
- To understand proportionality of similar polyhedra
- To become familiar with other polyhedra related to the icosahedron and dodecahedron

Prerequisites

The first part of this unit has no prerequisites. Familiarity with the concept of similarity and scaling is necessary for the second half, although it is also possible to do the unit while first studying similarity and scaling.

Notes

The word *polyhedron* was used in Unit 1 without a definition. There are many ways to define it. Since the Zome polyhedra are frameworks of edges and vertices, this book emphasizes the vertices and edges as determining the polyhedron. However, the word *polyhedron* suggests a definition that emphasizes the faces such as "a solid bounded by plane polygons." The names of specific polyhedra also emphasize the face. You can introduce the words *icosahedron* and *dodecahedron* as coming from the Greek *icosa* meaning 20, *dodeca* meaning 12, and *hedron* referring to the faces.

The polyhedral constructions in this unit involve building some scaffolding or intermediate structure as an aid to constructing the final form. In the end, the scaffolding is removed and just the intended polyhedron remains. If students see the pattern that develops after making some of the scaffolding, it is not necessary to make all the scaffolding; they can just continue the pattern to make the final form. If students get confused, they can go back to including the scaffolding.

Several of the polyhedra in this unit take considerable time and material. Student groups should make the first icosahedron and dodecahedron in different sizes. Then the groups' models can be saved and combined to build the larger structures in Activity 2.2.

Teacher Notes

2.1 Building and Counting

Instead of using the scaffolding strategy, you may suggest that students use what they learned in Unit 1 about regular polygons in order to build the faces.

The relationship between the numbers of faces, edges, and vertices will be discussed in Unit 6, and the answer to Question 6 will be explored further in Unit 9.

2.2 Scaling

This activity reviews the basic concepts of similarity and scaling, and previews the additive relationship between the Zome struts, which will be pursued in Unit 7.

Remind students that in making similar polyhedra, all lengths scale by the same amount. These include

- the distance between opposite faces
- the distance between opposite vertices
- the distance between opposite edges

Students can see all three of these scaling relationships at once with the construction of concentric dodecahedra.

In a $\mathbf{b}_1$ icosahedron, the distance between opposite vertices can be built with two $\mathbf{r}_1$s, and the distance between opposite edges can be built with a $\mathbf{b}_2$. However, there is no Zome length for the distance between the opposite faces of this icosahedron. You may discuss scaling by having students insert the appropriate struts in different-size icosahedra to show these lengths and their relationships.

The challenge is difficult. It is answered in Question 3. The scaled dodecahedra on the color insert display the answer for a similar question about the dodecahedra.

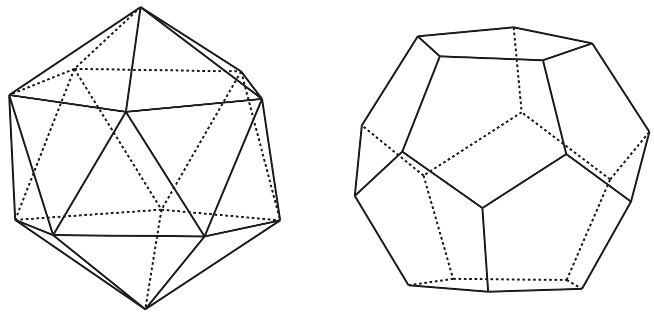
Explorations 2

After making any of these polyhedra, students should count the number of faces, edges, and vertices, and record the numbers for discussion in a future lesson. You may want to keep a class list of all the polyhedra students build, recording the name or description of the polyhedron, $F, E, V,$ and perhaps the name of the student builder.

2.1 Building and Counting

Challenge

The icosahedron consists of 20 equilateral triangles. The dodecahedron consists of 12 regular pentagons. Build them both.



Icosahedron and dodecahedron

1. If you did not succeed in building it on your own, here is a method to build the icosahedron using red struts as scaffolding. Make a three-dimensional starburst by putting red struts (all the same size) in all the pentagonal holes of one zomeball. Put another zomeball on the end of each red strut and connect them with blue struts. The blue is the icosahedron, so remove the red struts and the central ball.

   Q1 For the icosahedron, give the number of faces, edges, and vertices.

   Q2 How many edges does each face have? How many edges meet at each vertex?

   Q3 Notice how the icosahedron can be seen as an antiprism with two pyramids glued on. In how many different ways can you find a pentagonal antiprism in an icosahedron?

2. If you did not succeed in building it on your own, here is a method to build the dodecahedron using yellow struts as scaffolding. Make a three-dimensional starburst by putting yellow struts (all medium or all large) in all the triangular holes of one zomeball. Put another zomeball on the end of each yellow strut and connect them with blue struts. The blue is the dodecahedron, so remove the yellow struts and the central ball.

   Q4 For the dodecahedron, give the number of faces, edges, and vertices.

   Q5 How many edges does each face have? How many edges meet at each vertex?

   Q6 Compare the numbers of faces, edges, vertices, edges on each face, and edges that meet at each vertex in a dodecahedron to those in an icosahedron.

2.2 Scaling

Challenge

Which is taller: a b_3 icosahedron or a b_1 icosahedron stacked on top of a b_2 icosahedron?

1. Make a Zome model of two regular pentagons, such that pentagon *B*'s edge is twice as long as pentagon *A*'s. Include a diagonal in pentagon *A*.

   Q1 Predict the size of the diagonal in pentagon *B*.

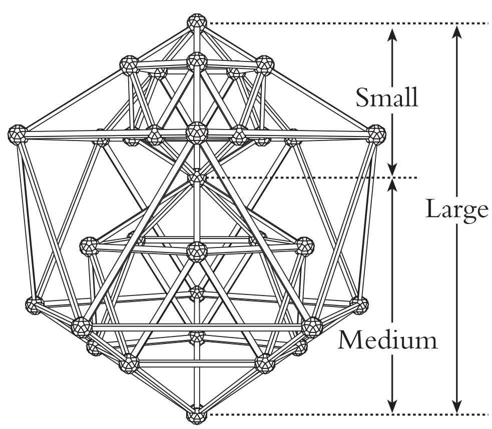
2. Check your prediction by building the diagonal.

   Q2 What is the scaling factor, or ratio of similarity, between pentagon *B* and pentagon *A*?

3. Combine two (or even three) dodecahedra with different edge lengths, smaller inside larger, with a common center. Use some radial yellow struts, as in the starburst, to connect them. Adding a ball at the very center and just two radial struts to the inner dodecahedron, construct and point out similar triangles.

If your eyes were at the center, your views of the dodecahedra would exactly overlap. The scaling factor (ratio of similarity) is b_3/b_2 or b_3/b_1 or b_2/b_1, depending on which pair of dodecahedra you are discussing.

4. Build small and medium icosahedra that touch at only one point, side-by-side inside a large icosahedron. (Hint: The b_1 and b_2 icosahedra have one vertex in common and are built on diametrically opposite sides of that vertex. At the vertices farthest from the common vertex, you need to extend the b_1 and b_2 triangles into b_3 triangles. To do this, notice that you can make a b_3 edge by connecting a b_1 and a b_2 in a straight line: $b_1 + b_2 = b_3$.)



Three icosahedra

   Q3 Hold your model three ways, standing it on a vertex, a face, and an edge. Describe how the sum of the small and medium add to the large for each distance between faces, between edges, and between vertices.

5. If you have time, you can make an analogous compound (small and medium sharing a vertex inside large) for the cube and the dodecahedron.

Explorations 2

Each of the following constructions leads you to another polyhedron related to the icosahedron and dodecahedron. After making any of these polyhedra, count the number of faces, edges, and vertices, and record the numbers for discussion in a future unit.

A. **Nonregular Icosahedron** Instead of using 30 blue struts, make an icosahedron using 10 blue, 10 red, and 10 yellow. This is a nonregular icosahedron, made of 20 triangles, but not equilateral triangles. It has the same topology as the regular icosahedron (the same number of edges, faces, and vertices, and they are connected in the same way) but different geometry (lengths and angles). There are several solutions.

B. **Nonregular Dodecahedron** Now make a nonregular dodecahedron from 10 blue, 10 red, and 10 yellow struts, topologically the same as a regular dodecahedron, but geometrically different. Again, there are several solutions.

C. **Elevated Dodecahedron** Take a regular dodecahedron and erect a blue pentagonal pyramid on each face. You need to use the next larger size strut (such as b_2 struts on a b_1 pentagon or b_3 struts on a b_2 pentagon), so the original dodecahedron must be small or medium, not large. You will be adding a new ball outside the middle of each face. When done, you have a nonconvex polyhedron consisting of 60 isosceles triangles.

D. **Concave Equilateral Deltahedron** Build a dodecahedron and erect a blue pentagonal pyramid on the inside of each face. You will be adding a new ball slightly inside the middle of each face. When done, you have a polyhedron consisting of 60 equilateral triangles. *Deltahedron* means made of triangles, not necessarily equilateral.

E. **Rhombic Triacontahedron 1** Build an icosahedron and erect a shallow red triangular pyramid on the outside of each face. You need to use the next smaller size strut (such as r_1 struts on a b_2 triangle), so the original icosahedron must be medium or large, not small. You will be adding a new ball slightly outside the middle of each face. Then remove the icosahedron. What is left will have rhombic faces.

F. **Rhombic Triacontahedron 2** Build a dodecahedron and erect a shallow red pentagonal pyramid on the outside of each face. You will be adding a new ball slightly outside the middle of each face. Then remove the dodecahedron. This gives the same result as the method in E!